An Introduction to Galileons

Shahab Shahidi

Damghan University

First School on Gravity and Particle Physics
5 Azar 1394

Introduction

- Beyond solar system, GR predictions become hopeless.
- At least at large scales, we need to modify GR; the IR modifications of gravity.
- And the simplest way is to add one extra dof to GR $\rightarrow$ scalar-tensor theories!
- This should be done in a healthy way.
- And one of its interesting candidates is Galileons.
- Galileons: first discovered in DGP; and the generalized to Horndeski theory.
- One of the well-motivated theories, which contain Galileons is the dRGT massive gravity theory.

$$\boxed{\text{DGP} \rightarrow \text{Galileons} \rightarrow \text{Horndeski} \rightarrow \text{dRGT MG}}$$

The DGP Model

- The DGP action (G Dvali, G Gabadadze and M Porrati, PLB 2000)

$$S = \int_{\text{bulk}} \mathrm{d}^5 x \left[\sqrt{-g_5} \frac{M_5^3}{4} R_5 + \delta(y) \sqrt{-g_4} \left(\frac{M_{\text{Pl}}^2}{2} R_4 + \mathcal{L}_{\text{matter}}[g] \right) \right].$$

- The bulk and brane Planck masses define a crossover scale

$$2 r_c = \frac{M_{\text{Pl}}^2}{M_5^3} \equiv \frac{1}{m_0}.$$

- All standard model particles are supposed to live in 4D, and gravity can propagate to 5D space.
- The bulk is assumed to be $\mathbb{Z}_2$-symmetric wrt the brane.
- Solving the Hierarchy problem!

The DGP Model

- 5 helicity dof of a massless graviton in 5D $\rightarrow$ 5 dof of a massive graviton in 4D!
- In the decoupling limit the action for DGP gravity reduces to

$$\mathcal{L}_{\mathrm{DL,DGP}} = \frac{1}{8} h^{\mu\nu} \Box \left(h_{\mu\nu} - \frac{1}{2} h \eta_{\mu\nu} \right) - \frac{1}{4} N^{\mu} \sqrt{-\Box} N_{\mu} + \frac{3}{2} \pi \Box \pi + \frac{1}{2\Lambda_3^3} (\partial \pi)^2 \Box \pi .$$

  where $\Lambda_3 = (m_0^2 M_{Pl})^{1/3}$. (M Luty, et al., JHEP 2003)
- Decoupling limit is defined as

$$M_{Pl} \rightarrow \infty, \qquad m_0 \rightarrow 0, \quad \text{but} \quad \Lambda_3 \text{ remains finite!}$$

Some properties

- Now, consider the following action in 4D

$$S = \int d^4x \left[\frac{M_{\rm Pl}^2}{2} \sqrt{-g} R - \frac{1}{2} (\partial\pi)^2 + \frac{1}{2\Lambda_3^3} (\partial\pi)^2 \Box\pi \right]$$

- The above action is invariant under the Galilean shift symmetry

$$\partial_\mu \pi \to \partial_\mu \pi + c_\mu$$

- The equation of motion for the scalar field π is at most second order in time derivatives. (No Ostrogradski instability.)

$$\mathcal{E}_\pi = \Box\pi - \frac{1}{\Lambda_3^3} \left[(\Box\pi)^2 - (\partial_\mu \partial_\nu \pi)^2 \right] .$$

Some properties

- The Lagrangian changes by a total derivative under Galilean transformation.
- On the other hand, the Feynman vertex derived from the scalar self interaction term is invariant under the Galilean transformations.
- Any Feynman diagram in this theory can only give rise to counter-terms in the Lagrangian that would be invariant under Galilean transformations.
- The self-interaction term itself is not among such terms. Therefore, it does not get renormalized.
- The non-renormalization theorem of the DGP model.

Some properties

- The one-loop effective action for this theory can be written as

$$\Gamma^{1-loop} = \sum_m \left[a_m \Lambda^4 + b_m \Lambda^2 \partial^2 + (c_m \ln \Lambda + I_m)\partial^4 \right] \left(\frac{\partial\partial\pi}{\Lambda_3^3} \right)^m .$$

- Does not renormalize the new vertex. (M Luty, et al., JHEP 2003)
- The above Lagrangian possesses superluminal group velocities in the vicinity of massive object!

Galileons!

- Because the scalar field respects the Galilean symmetry, we call it, Galileon!
- Question: Are there any other interaction term with these properties?
- Answer: Yes! We just need to make the bulk action richer!
- Adding Gauss-Bonnet term to the bulk action!

(C de Rham & A Tolley, JCAP 2010)

- OR you can just search for such terms with try and error!

(A Nicolis, et al., PRD 2009)

- The generalized action can be written as

$$S = \int d^4x \left[\frac{1}{2} M_{\rm Pl}^2 \sqrt{-g} R + \frac{1}{2} h_{\mu\nu} T^{\mu\nu} + \mathcal{L}_\pi + \pi \, T^\mu{}_\mu \right]$$

The structure of $\mathcal{L}_\pi$

- Some definitions:

$$\Pi^{\mu}{}_{\nu} \equiv \partial^\mu \partial_\nu \pi$$

- The brackets $[\ldots]$ stand for the trace operator, and '$\cdot$' stands for the standard Lorentz-invariant contraction of indices.
- For example

$$[\Pi]\,\partial\pi \cdot \partial\pi \equiv \Box\pi\,\partial_\mu\pi\partial^\mu\pi$$

The structure of $\mathcal{L}_\pi$

- The only terms which satisfy DGP conditions and not a total derivative in $4D$ are (A Nicolis, et al., PRD 2009)

$$
\begin{aligned}
\mathcal{L}_1 =& \pi \\
\mathcal{L}_2 =& \partial\pi \cdot \partial\pi \\
\mathcal{L}_3 =& [\Pi]\,\partial\pi \cdot \partial\pi \\
\mathcal{L}_4 =& [\Pi]^2\,\partial\pi \cdot \partial\pi - 2\,[\Pi]\,\partial\pi \cdot \Pi \cdot \partial\pi - [\Pi^2]\,\partial\pi \cdot \partial\pi \\
& + 2\,\partial\pi \cdot \Pi^2 \cdot \partial\pi, \\
\mathcal{L}_5 =& [\Pi]^3\,\partial\pi \cdot \partial\pi - 3[\Pi]^2\,\partial\pi \cdot \Pi \cdot \partial\pi - 3[\Pi][\Pi^2]\,\partial\pi \cdot \partial\pi \\
& + 6[\Pi]\,\partial\pi \cdot \Pi^2 \cdot \partial\pi + 2[\Pi^3]\,\partial\pi \cdot \partial\pi + 3[\Pi^2]\,\partial\pi \cdot \Pi \cdot \partial\pi \\
& - 6\,\partial\pi \cdot \Pi^3 \cdot \partial\pi.
\end{aligned}
$$

- Above 4D, we have higher order Galileon interactions!

The structure of $\mathcal{L}_\pi$

- Equations of motion associated to the above Lagrangians are

$$\begin{aligned}
\mathcal{E}_1 &= 1 \\
\mathcal{E}_2 &= \Box\pi \\
\mathcal{E}_3 &= (\Box\pi)^2 - (\partial_\mu\partial_\nu\pi)^2 \\
\mathcal{E}_4 &= (\Box\pi)^3 - 3\,\Box\pi(\partial_\mu\partial_\nu\pi)^2 + 2(\partial_\mu\partial_\nu\pi)^3 \\
\mathcal{E}_5 &= (\Box\pi)^4 - 6(\Box\pi)^2(\partial_\mu\partial_\nu\pi)^2 + 8\,\Box\pi(\partial_\mu\partial_\nu\pi)^3 \\
&\quad + 3\left[(\partial_\mu\partial_\nu\pi)^2\right]^2 - 6(\partial_\mu\partial_\nu\pi)^4
\end{aligned}$$

- The Galileon equation of motion becomes

$$\mathcal{E} \equiv \frac{\delta\mathcal{L}_\pi}{\delta\pi} = \sum_{i=1}^{5} c_i\,\mathcal{E}_i = -T.$$

Properties of Galileons

- Galileons, inherit all the DGP-interaction term:
  - They have higher order derivative terms in the action.
  - But they produce at most second order field equations.
  - They respect the Galileon symmetry.
  - They respect the non-renormalization theorem.
  - They possess superluminal solutions.

Covariant Galileons

- Up to this point all calculations have been done on top of flat space.
- Question: What happens in curved space-times?
- One can replace all partial derivatives with covariant derivatives!
- The first three terms produce second order field equations again.
- But the fourth and fifth terms become ghosty!

- The resulting Lagrangians become

$$
\begin{aligned}
\mathcal{L}_4 &= (\Box\pi)^2 \left(\pi_{;\mu}\, \pi^{;\mu}\right) - 2\, (\Box\pi) \left(\pi_{;\mu}\, \pi^{;\mu\nu}\, \pi_{;\nu}\right) \\
&\quad - \left(\pi_{;\mu\nu}\, \pi^{;\mu\nu}\right) \left(\pi_{;\rho}\, \pi^{;\rho}\right) + 2 \left(\pi_{;\mu}\pi^{;\mu\nu}\, \pi_{;\nu\rho}\, \pi^{;\rho}\right), \\
\mathcal{L}_5 &= (\Box\pi)^3 \left(\pi_{;\mu}\, \pi^{;\mu}\right) \\
&\quad - 3\, (\Box\pi)^2 \left(\pi_{;\mu}\, \pi^{;\mu\nu}\, \pi_{;\nu}\right) - 3\, (\Box\pi) \left(\pi_{;\mu\nu}\, \pi^{;\mu\nu}\right) \left(\pi_{;\rho}\, \pi^{;\rho}\right) \\
&\quad + 6\, (\Box\pi) \left(\pi_{;\mu}\pi^{;\mu\nu}\, \pi_{;\nu\rho}\, \pi^{;\rho}\right) + 2 \left(\pi_{;\mu}{}^{\nu}\, \pi_{;\nu}{}^{\rho}\, \pi_{;\rho}{}^{\mu}\right) \left(\pi_{;\lambda}\, \pi^{;\lambda}\right) \\
&\quad + 3 \left(\pi_{;\mu\nu}\, \pi^{;\mu\nu}\right) \left(\pi_{;\rho}\, \pi^{;\rho\lambda}\, \pi_{;\lambda}\right) - 6 \left(\pi_{;\mu}\, \pi^{;\mu\nu}\, \pi_{;\nu\rho}\, \pi^{;\rho\lambda}\, \pi_{;\lambda}\right)
\end{aligned}
$$

- Consider only the first term!

- The equation of motion for the π field is

$$
\begin{aligned}
\mathcal{E}_4 \equiv\ & 2\left(\pi_{;\mu}\,\pi^{;\mu}\right)\left(\pi_{;\nu}{}^{\nu}{}_{\rho}{}^{\rho} - \pi_{;\nu\rho}{}^{\nu\rho}\right) \\
& + 2\,\pi^{;\mu}\,\pi^{;\nu}\left(2\,\pi_{;\mu\rho\nu}{}^{\rho} - \pi_{;\mu\nu\rho}{}^{\rho} - \pi_{;\rho}{}^{\rho}{}_{\mu\nu}\right) \\
& + 10\left(\Box\pi\right)\pi^{;\mu}\left(\pi_{;\mu\nu}{}^{\nu} - \pi_{;\nu}{}^{\nu}{}_{\mu}\right) + 12\,\pi_{;\mu}\,\pi^{;\mu\nu}\left(\pi_{;\rho}{}^{\rho}{}_{\nu} - \pi_{;\nu\rho}{}^{\rho}\right) \\
& + 8\,\pi^{;\mu}\,\pi^{;\nu\rho}\left(\pi_{;\nu\rho\mu} - \pi_{;\mu\nu\rho}\right) \\
& - 4\left(\Box\pi\right)^3 - 8\left(\pi_{;\mu}{}^{\nu}\,\pi_{;\nu}{}^{\rho}\,\pi_{;\rho}{}^{\mu}\right) + 12\left(\Box\pi\right)\left(\pi_{;\mu\nu}\,\pi^{;\mu\nu}\right),
\end{aligned}
$$

- It contains third and fourth-order derivatives.
- Ghost backs to the theory.

- Commuting the derivatives one can find

$$\begin{aligned}\mathcal{E}_4 = & -4\,(\Box\pi)^3 - 8\left(\pi_{;\mu}{}^{\nu}\,\pi_{;\nu}{}^{\rho}\,\pi_{;\rho}{}^{\mu}\right) + 12\,(\Box\pi)\,(\pi_{;\mu\nu}\pi^{;\mu\nu}) \\ & - (\pi_{;\mu}\,\pi^{;\mu})\,(\pi_{;\nu}\,R^{;\nu}) + 2\,(\pi_{;\mu}\,\pi_{;\nu}\,\pi_{;\rho}R^{\mu\nu;\rho}) \\ & + 10\,(\Box\pi)\,(\pi_{;\mu}\,R^{\mu\nu}\,\pi_{;\nu}) - 8\,(\pi_{;\mu}\,\pi^{;\mu\nu}\,R_{\nu\rho}\,\pi^{;\rho}) \\ & - 2\,(\pi_{;\mu}\,\pi^{;\mu})\,(\pi_{;\nu\rho}\,R^{\nu\rho}) - 8\,(\pi_{;\mu}\,\pi_{;\nu}\,\pi_{;\rho\sigma}\,R^{\mu\rho\nu\sigma})\,.\end{aligned}$$

- The higher-order derivatives vanishes in flat space-time.
- Non-minimal couplings to the metric can eliminate the higher-order derivative terms.

Covariant Galileons

- The only possible nonvanishing terms made of contractions of four gradiants of π and one curvature tensor are

$$\mathcal{L}_{4,1} = (\pi_{;\mu}\,\pi^{;\mu})\,(\pi_{;\nu}\,\pi^{;\nu})\,R,$$
$$\mathcal{L}_{4,2} = \left(\pi_{;\lambda}\,\pi^{;\lambda}\right)(\pi_{;\mu}\,R^{\mu\nu}\,\pi_{;\nu})\,.$$

- The combination $\mathcal{L}_{4,2} - \frac{1}{2}\mathcal{L}_{4,1}$ eliminates all higher order derivatives in the π equation of motion.

- If we add to

$$S_4 = \int d^4x\sqrt{-g}\,\mathcal{L}_4$$

the new action

$$S_4^{\text{nonmin}} \equiv \int d^4x\sqrt{-g}\left(\pi_{;\lambda}\,\pi^{;\lambda}\right)\pi_{;\mu}\left[R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R\right]\pi_{;\nu}$$
$$= \int d^4x\sqrt{-g}\left(\pi_{;\lambda}\,\pi^{;\lambda}\right)\left(\pi_{;\mu}\,G^{\mu\nu}\,\pi_{;\nu}\right),$$

all higher order derivatives vanishes from the field equations.

- The new equation of motion

$$\begin{aligned}\mathcal{E}_4' = & -4\,(\Box\pi)^3 - 8\left(\pi_{;\mu}{}^{\nu}\,\pi_{;\nu}{}^{\rho}\,\pi_{;\rho}{}^{\mu}\right) + 12\,(\Box\pi)\,(\pi_{;\mu\nu}\pi^{;\mu\nu}) \\ & + 2\,(\Box\pi)\,(\pi_{;\mu}\,\pi^{;\mu})\,R + 4\,(\pi_{;\mu}\,\pi^{;\mu\nu}\,\pi_{;\nu})\,R \\ & + 8\,(\Box\pi)\,(\pi_{;\mu}\,R^{\mu\nu}\,\pi_{;\nu}) - 4\left(\pi_{;\lambda}\,\pi^{;\lambda}\right)(\pi_{;\mu\nu}\,R^{\mu\nu}) \\ & - 16\,(\pi_{;\mu}\,\pi^{;\mu\nu}\,R_{\nu\rho}\,\pi^{;\rho}) - 8\,(\pi_{;\mu}\,\pi_{;\nu}\,\pi_{;\rho\sigma}\,R^{\mu\rho\nu\sigma})\,.\end{aligned}$$

- Ostrogradski instability is absent here.
- Involves first-order derivatives of π.
- This breaks the "Galilean" symmetry.
- Exorcising the ghost with the price of missing the Galileon symmetry.

The Horndeski theory

- The full action can be written in a simpler form

$$S_4' = \int d^4x\sqrt{-g}\left(\pi_{;\lambda}\,\pi^{;\lambda}\right)\left[2\left(\Box\pi\right)^2 - 2\left(\pi_{;\mu\nu}\,\pi^{;\mu\nu}\right) - \frac{1}{2}\left(\pi_{;\mu}\,\pi^{;\mu}\right)R\right].$$

- The fifth term can be modified to

$$S_5' = \frac{5}{2}\int d^4x\sqrt{-g}\left(\pi_{;\lambda}\,\pi^{;\lambda}\right)\Bigg[\left(\Box\pi\right)^3 - 3\left(\Box\pi\right)\left(\pi_{;\mu\nu}\,\pi^{;\mu\nu}\right) + 2\left(\pi_{;\mu}{}^{\nu}\,\pi_{;\nu}{}^{\rho}\,\pi_{;\rho}{}^{\mu}\right) - 6\left(\pi_{;\mu}\,\pi^{;\mu\nu}\,G_{\nu\rho}\,\pi^{;\rho}\right)\Bigg].$$

- This has been done 40 years ago by Horndeski, who had searched for a theory of gravity+scalar fields with no higher derivative instabilities.
- So, we call these covariant Galileons: The Horndeski theory.

International Journal of Theoretical Physics, Vol. 10, No. 6 (1974), pp. 363–384

Second-Order Scalar-Tensor Field Equations in a Four-Dimensional Space

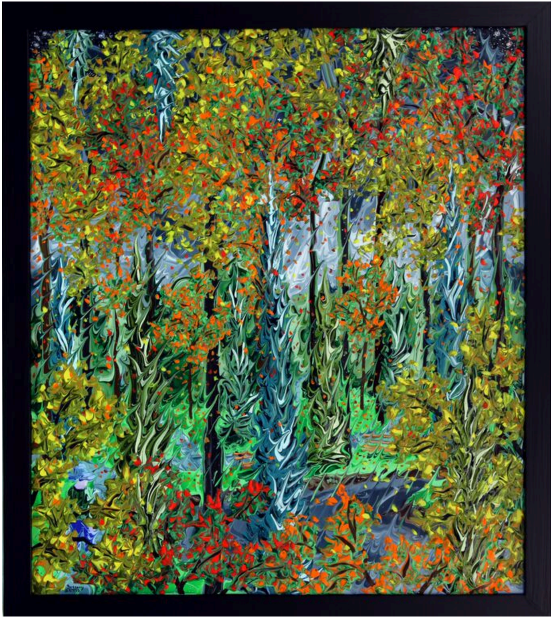
GREGORY WALTER HORNDESKI

Department of Applied Mathematics, University of Waterloo, Waterloo, Ontario, Canada

Received: 10 *July* 1973

Abstract

Lagrange scalar densities which are concomitants of a pseudo–Riemannian metric-tensor, a scalar field and their derivatives of arbitrary order are considered. The most general second-order Euler–Lagrange tensors derivable from such a Lagrangian in a four-dimensional space are constructed, and it is shown that these Euler–Lagrange tensors may be obtained from a Lagrangian which is at most of second order in the derivatives of the field functions.



Generalized Galileons

- This is not the end! We can generalize these Lagrangians even more!
- Beyond Horndeski (J Gleyzes, et al., PRL 2015)

$$
\begin{aligned}
L_2 &\equiv G_2(\pi, X), \\
L_3 &\equiv G_3(\pi, X)\,\Box\pi, \\
L_4 &\equiv G_4(\pi, X)R - 2\,G_{4,X}(\pi, X)(\Box\pi^2 - \Pi^{\mu\nu}\Pi_{\mu\nu}) \\
&\quad + H_4(\pi, X)\epsilon^{\mu\nu\rho}{}_{\sigma}\,\epsilon^{\mu'\nu'\rho'\sigma}\pi_\mu\pi_{\mu'}\Pi_{\nu\nu'}\Pi_{\rho\rho'}, \\
L_5 &\equiv G_5(\pi, X)\,G_{\mu\nu}\Pi^{\mu\nu} \\
&\quad + \frac{1}{3}G_{5,X}(\pi, X)(\Box\pi^3 - 3\,\Box\pi\,\Pi_{\mu\nu}\Pi^{\mu\nu} + 2\,\Pi_{\mu\nu}\Pi^{\mu\sigma}\Pi^{\nu}{}_{\sigma}) \\
&\quad + H_5(\pi, X)\epsilon^{\mu\nu\rho\sigma}\epsilon^{\mu'\nu'\rho'\sigma'}\pi_\mu\pi_{\mu'}\Pi_{\nu\nu'}\Pi_{\rho\rho'}\Pi_{\sigma\sigma'},
\end{aligned}
$$

- where $\pi_\mu = \nabla_\mu\pi$ and $\Pi_{\mu\nu} = \nabla_\mu\nabla_\nu\pi$ and $X = \nabla_\mu\pi\nabla^\mu\pi$.

dRGT massive gravity

- There is an interesting example of appearance of Galileons in gravity, dRGT massive gravity. (C de Rham, et al. PRL 2010)

$$S = M_p^2 \int d^4x \sqrt{-g}\, R(g) - 2M_p^2 m^2 \int d^4x \sqrt{-g} \left[\mathcal{U}_2(\mathcal{K}) + \alpha_3 \mathcal{U}_3(\mathcal{K}) + \alpha_4 \mathcal{U}_4(\mathcal{K}) \right].$$

where

$$\mathcal{U}_2(\mathcal{K}) = [\mathcal{K}]^2 - [\mathcal{K}^2],$$
$$\mathcal{U}_3(\mathcal{K}) = [\mathcal{K}]^3 - 3[\mathcal{K}][\mathcal{K}^2] + 2[\mathcal{K}^3],$$
$$\mathcal{U}_4(\mathcal{K}) = [\mathcal{K}]^4 - 6[\mathcal{K}^2][\mathcal{K}]^2 + 8[\mathcal{K}^3][\mathcal{K}] + 3[\mathcal{K}^2]^2 - 6[\mathcal{K}^4],$$

and

$$\mathcal{K}^\mu_\nu(g, \phi^a) = \delta^\mu_\nu - \left(\sqrt{g^{-1} f}\right)^\mu{}_\nu$$

- $f_{\mu\nu}$ is the fiducial metric: $f_{\mu\nu} = \eta_{ab}\partial_\mu \phi^a \partial_\nu \phi^b$
- ϕ^a are four Stuckelberg fields.

- In the decoupling limit one can find (C de Rham & G Gabadadze, PRD 2010)

$$\mathcal{L}_{DL} \ni -\frac{1}{4}\left[h^{\mu\nu}\mathcal{E}^{\alpha\beta}_{\mu\nu}h_{\alpha\beta} + \sum_{n=2}^{5}\frac{c_n}{\Lambda_3^{3(n-2)}}\mathcal{L}^{(n)}_{(Gal)}\right].$$

- The helicity-0 of the massive graviton behaves like a Galileon!
- dRGT does not have a flat FRW solution! (G D'Amico, et al., PRD 2011)
- Only has an open FRW solution, which is self-accelerating!
- Perturbations on top of this solution. the helicity-0 dof of massive graviton becomes unstable! (A Gumrukcuoglu, et al., JCAP 2011)
- These problems can be solved by adding a scalar field! The quasi-dilaton. (see next talk!)

Thanks for your attention!