Axions via Cartan-Gauss-Bonnet

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The Weyl theory

- One of the first extensions of GR was introduced in 1918 by Hermann Weyl. (Sitzungsber. Preuss. Akad. Wiss. 465 (1918) 14)
- His purpose was unification of the long range forces i.e. Gravity and Electromagnetism.
- The connection in Weyl theory is no longer metric compatible

$$\tilde{\nabla}_\mu g_{\nu\rho} = 2w_\mu g_{\nu\rho},$$

- This leads to change in the length of a vector field during parallel transportation.
- The above equation with the assumption $\Gamma^\lambda{}_{\mu\nu} = \Gamma^\lambda{}_{\nu\mu}$ leads to

$$\Gamma^\lambda_{\mu\nu} = \left\{ {\lambda \atop \mu\nu} \right\} + g_{\mu\nu}w^\lambda - \delta^\lambda_\nu w_\mu - \delta^\lambda_\mu w_\nu.$$

The Cartan geometry

- In Cartan geometry, the connection is no longer symmetric.
- The antisymmetric part of the connection is the torsion

$$T^{\lambda}_{\ \mu\nu} = \frac{1}{2}\left(\Gamma^{\lambda}_{\ \mu\nu} - \Gamma^{\lambda}_{\ \nu\mu}\right).$$

- The notion of an asymmetric affine connection was mentioned by Eddington in 1921.
- Torsion was introduced by Elie Cartan in 1922.

(C. R. Acad. Sci. (Paris) 174, (1922) 593)

Cartan geometry

- In the Riemann-Cartan space we have

$$\nabla_\alpha g_{\mu\nu} = 0$$

so, one can obtain

$$\Gamma^\rho_{\mu\nu} = \left\{ {}^{\rho}_{\mu\nu} \right\} + C^\rho{}_{\mu\nu}.$$

where $C^\rho{}_{\mu\nu}$ is the contortion tensor and defined as

$$C^\rho{}_{\mu\nu} = T^\rho{}_{\mu\nu} - g^{\rho\beta} g_{\sigma\mu} T^\sigma{}_{\beta\nu} - g^{\rho\beta} g_{\sigma\nu} T^\sigma{}_{\beta\mu}.$$

The Gauss-Bonnet (GB) gravity

- Kretschmann in 1917 introduced the action of the form

(Ann. Phys. (Leipzig), 53 (1917) 575)

$$S_K = \int d^4x \sqrt{-g} R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma},$$

- The above action has higher than second time derivatives in the action, and therefore it is unstable!
- The unique combination of curvature tensor which is second order in curvature and has at most second order time derivatives is

$$S_{GB} = \int d^4x \sqrt{-g} (R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 4R_{\mu\nu} R^{\mu\nu} + R^2),$$

known as Gauss-Bonnet term.
- In 4D, the above combination becomes a total derivative, and does not contribute to the action.

The Weyl-GB gravity

- Consider the action

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{2\kappa^2}K + \mathcal{L}_G\right],$$

with

$$\mathcal{L}_G = \alpha_1 K^{\alpha\beta\gamma\delta}K_{\alpha\beta\gamma\delta} + \alpha_2 K^{\alpha\beta\gamma\delta}K_{\gamma\delta\alpha\beta} - \alpha_3 K^{\alpha\beta\gamma\delta}K_{\beta\alpha\gamma\delta}$$
$$- 4(\beta_1 \mathcal{K}_{\beta\gamma}\mathcal{K}^{\beta\gamma} + \beta_2 \mathcal{K}_{\beta\gamma}\mathcal{K}^{\gamma\beta} + \beta_3 K_{\alpha\beta}K^{\alpha\beta} + \beta_4 K_{\alpha\beta}\mathcal{K}^{\alpha\beta})$$
$$+ K^2.$$

where $K_{\mu\nu} \equiv K^{\alpha}{}_{\mu\alpha\nu}$ and $\mathcal{K}_{\mu\nu} \equiv K^{\alpha}{}_{\alpha\mu\nu}$.

- We should impose the constraints:

$$\alpha_1 + \alpha_2 + \alpha_3 = 1,$$

$$\beta_1 + \beta_2 = 1.$$

The Weyl-GB gravity

- Using the definition of non-metricity, the action leads to

(Jimenez and Koivisto 2014)

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{2\kappa^2}R - \frac{1}{2}m^2 w_\mu w^\mu - \frac{1}{4}\alpha W_{\mu\nu}W^{\mu\nu}\right],$$

- With α is a constant and a combination of the coupling in the GB Lagrangian.
- Also $W_{\mu\nu} = \partial_\mu w_\nu - \partial_\nu w_\mu$.
- This is the Einstein-Proca system.

- There are two independent combinations of the curvature tensor in this case

$$K_{\lambda\mu\nu\sigma}K^{\lambda\mu\nu\sigma}, \qquad K_{\lambda\mu\nu\sigma}K^{\nu\sigma\lambda\mu}.$$

- And two independent combinations for the contracted curvature tensor

$$K_{\mu\nu}K^{\mu\nu}, \qquad K_{\mu\nu}K^{\nu\mu}.$$

- So, one can write

$$\begin{aligned}\mathcal{L}_G =& \alpha K^{\alpha\beta\gamma\delta}K_{\alpha\beta\gamma\delta} + (1-\alpha)K^{\alpha\beta\gamma\delta}K_{\gamma\delta\alpha\beta} \\ &- 4\beta K_{\beta\gamma}K^{\beta\gamma} - 4(1-\beta)K_{\beta\gamma}K^{\gamma\beta} + K^2.\end{aligned}$$

- Let's decompose the torsion tensor as

$$T_{\mu\nu\rho} = \frac{2}{3}(t_{\mu\nu\rho} - t_{\mu\rho\nu}) + \frac{1}{3}(\hat{Q}_\nu g_{\mu\rho} - \hat{Q}_\rho g_{\mu\nu}) + \epsilon_{\mu\nu\rho\sigma} S^\sigma,$$

- with

$$t_{\mu\nu\rho} + t_{\nu\rho\mu} + t_{\rho\mu\nu} = 0, \quad g_{\mu\nu} t^{\mu\nu\rho} = 0 = g_{\mu\rho} t^{\mu\nu\rho}.$$

- Obtaining the following decomposition for contortion

$$C_{\rho\mu\nu} = \frac{4}{3}(t_{\mu\nu\rho} - t_{\rho\nu\mu}) + \frac{2}{3}(\hat{Q}_\mu g_{\nu\rho} - \hat{Q}_\rho g_{\mu\nu}) + \epsilon_{\rho\mu\nu\sigma} S^\sigma,$$

- Let's assume that $t_{\mu\nu\rho} = 0$.

The Cartan-GB gravity

- The pure Q_μ part is equivalent to the Weyl-Gauss-Bonnet theory; becomes Einstein-Proca system.
- The pure S_μ part is also equivalent to Weyl-Gauss-Bonnet theory, after removing the unhealthy d.o.f. This gives us $\alpha = 0$.
- The mixed term can be written as

$$\epsilon^{\mu\nu\rho\sigma} Q_{\mu\nu} S_{\rho\sigma}.$$

- This term is a total derivative

$$\sqrt{-g}\epsilon^{\mu\nu\rho\sigma} Q_{\mu\nu} S_{\rho\sigma} = \partial_\mu \left(\epsilon^{\mu\nu\rho\sigma} Q_\nu S_{\rho\sigma} \right).$$

The Axion-Cartan-GB gravity

- Let's try to keep the mixed term, by couple the torsion to a scalar field via

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{2\kappa^2}K + \rho\bar{\mathcal{L}}_G - \frac{1}{2}\partial_\mu\beta\partial^\mu\beta - V(\beta)\right],$$

with modified Guass-Bonnet term defined as

$$\begin{aligned}\bar{\mathcal{L}}_G =& \alpha K^{\alpha\beta\gamma\delta}K_{\alpha\beta\gamma\delta} + (1-\alpha)K^{\alpha\beta\gamma\delta}K_{\gamma\delta\alpha\beta} \\ & - 4\tilde{\beta}(x)K_{\beta\gamma}K^{\beta\gamma} - 4(1-\tilde{\beta}(x))K_{\beta\gamma}K^{\gamma\beta} + K^2.\end{aligned}$$

- We have defined $\beta \equiv M\tilde{\beta}$ with some constant M with dimension of mass.
- ρ is a coupling constant.

- After substituting the torsion, one obtains the action

$$S = \int d^4x\sqrt{-g}\Bigg[\frac{1}{2\kappa^2}R - \frac{1}{2}m^2 Q_\mu Q^\mu - \frac{1}{4}(1+32\tilde{\phi})Q_{\mu\nu}Q^{\mu\nu} - \frac{1}{2}m^2 S_\mu S^\mu - \frac{1}{4}(1+32\tilde{\phi})S_{\mu\nu}S^{\mu\nu} - 8\tilde{\phi}(x)\epsilon^{\mu\nu\rho\sigma}Q_{\mu\nu}S_{\rho\sigma} - \frac{1}{2}\partial_\mu\phi(x)\partial^\mu\phi(x) - V(\phi)\Bigg],$$

with $\phi = \beta - \frac{1}{32}M$, and again $\tilde{\phi} \equiv \phi/M$

- Assuming $Q_\mu = S_\mu$ gives

$$S = \int d^4x\sqrt{-g}\Big[\frac{1}{2\kappa^2}R - \frac{1}{2}m^2 A_\mu A^\mu - \frac{1}{4}(1+32\tilde{\phi})F_{\mu\nu}F^{\mu\nu} - 4\tilde{\phi}(x)\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} - \frac{1}{2}\partial_\mu\phi(x)\partial^\mu\phi(x) - V(\phi)\Big],$$

- where $A_\mu = \sqrt{2}Q_\mu$, $m^2 = 6/(\rho\kappa^2)$ and $F_{\mu\nu}$ is its field strength defined as $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

Cosmological implications

- Use $A_\mu = [A(t), B(t), B(t), B(t)]$ in the flat FRW ansatz.
- One obtains $A(t) = 0$ from its field equation.
- The dynamical equations become

$$3\left((32\phi+1)\left(\dot{b}+bh\right)^2+\mu^2b^2-4h^2\right)+2V+\dot{\phi}^2=0,$$

$$48\left(\dot{b}+bh\right)^2-3h\dot{\phi}-V'-\ddot{\phi}=0,$$

$$(32\phi+1)\ddot{b}+\dot{b}\left(h(96\phi+3)+32\dot{\phi}\right)$$
$$+b\left((32\phi+1)\dot{h}+32h\dot{\phi}+h^2(64\phi+2)+\mu^2\right)=0,$$

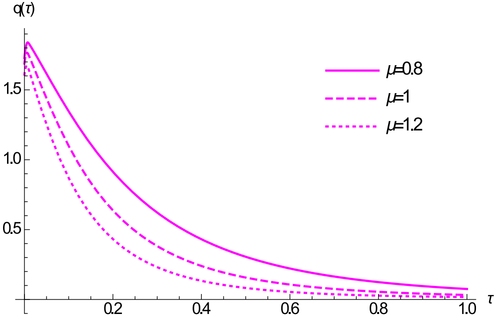
where we have defined

$$\tau=H_0t,\quad h=\frac{H}{H_0},\quad \mu=\frac{m}{H_0},\quad b=\frac{B}{a}.$$

- We have assumed that $M=1$ for simplicity.

The case $V = 0$

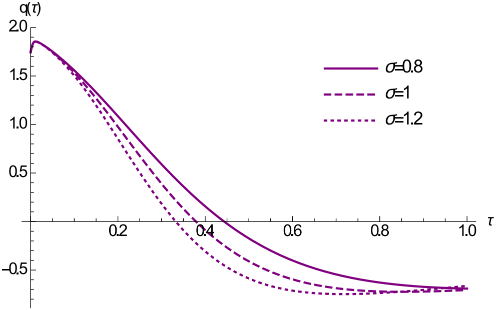
- The deceleration parameter is



- Larger the mass of the gauge field results in a less decelerated universe.
- The theory without potential can not explain the accelerated expanding universe.

The case $V = \frac{1}{2}m_\phi^2\phi^2$

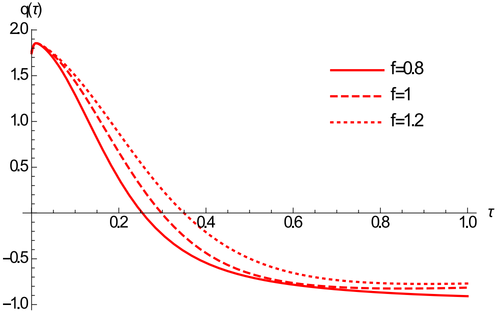
- Consider a quadratic potential $V = \frac{1}{2}m_\phi^2\phi^2$.
- Introduce $\sigma = \frac{m_\phi}{H_0}$.
- The deceleration parameter is



- For Larger values of the axion mass, the universe enters to the accelerated expanding phase sooner.

The $V = V_0(1 - \cos(\frac{\phi}{f}))$

- The deceleration parameter is of the form



- For smaller f, the universe enters to the accelerated expanding phase sooner.

Summary

- We have considered the Gauss-Bonnet theory in the Cartan spacetime.
- The role of the trace part of the torsion is equivalent to the Weyl vector. Both leads to the Einstein-Proca system.
- The S part in general introduces an unwanted d.o.f. But tunning the parameters, can avoid this d.o.f. The resulting Lagrangian is again similar to the Weyl-Gauss-Bonnet.
- A theory with Weyl vector and the trace of torsion tensor leads to the Einstein-Maxwell-Proca system.
- Coupling between Q_μ and S^μ produces an interesting term.
- To produce an accelerating universe, the theory needs a potential for the scalar field.
- The cosmological perturbation analysis gives us more information about the effects of extra term in the action.

Thanks for your attention!