$$U(x,y,z) = \iint F(\nu_x,\nu_y)\exp[-j2\pi(\nu_x x+\nu_y y)]\exp(\ldots)$$

$$k_z = \sqrt{k^2-k_x^2-k_y^2} = 2\pi\sqrt{\frac{1}{\lambda^2}-\frac{1}{\lambda_x^2}-\frac{1}{\lambda_y^2}}$$

→ An incident plane wave is decomposed into many plane waves, each traveling at angles $\theta_x = \sin^{-1}(\lambda\nu_x)$, $\theta_y = \sin^{-1}(\lambda\nu_y)$, with a complex envelope $F(\nu_x,\nu_y)$, the Fourier transform of $f(x,y)$.

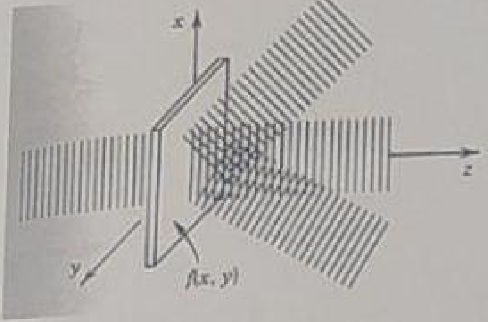


Figure 4.1-3 A thin optical element of amplitude transmittance $f(x,y)$ decomposes an incident plane wave into many plane waves. The plane wave traveling at the angles $\theta_x = \sin^{-1}\lambda\nu_x$ and $\theta_y = \sin^{-1}\lambda\nu_y$ has a complex envelope $F(\nu_x,\nu_y)$, the Fourier transform of $f(x,y)$.

Example 4.1-2, Imaging

$$t(x,y) = \exp\left(j\frac{\pi x^2}{\lambda f}\right) = \exp(-j2\pi\varphi(x,y))$$

$$\varphi(x,y) = -\frac{x^2}{2\lambda f}$$

Compare to earlier: $\varphi(x,y) \leftrightarrow \nu_x x + \nu_y y$

Now ν_x varies with $x \Rightarrow \nu_x = \frac{\partial\varphi(x,y)}{\partial x} = -\frac{x}{\lambda f}$

$$\Rightarrow \theta_x = \sin^{-1}\left(-\frac{x}{f}\right)$$

→ A cylindrical lens with focal length f

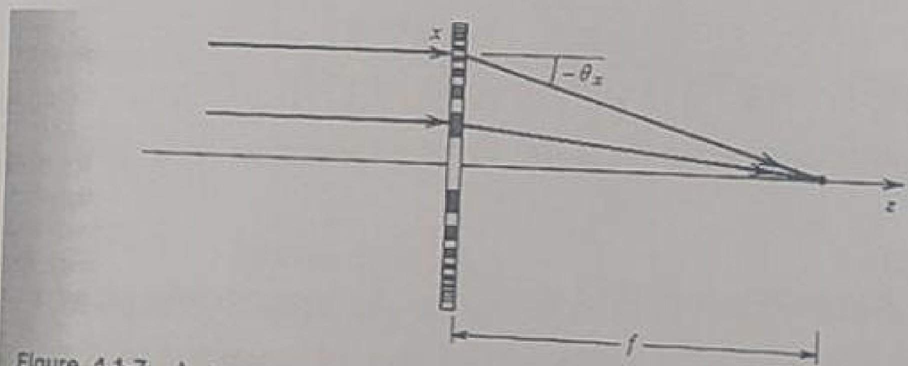


Figure 4.1-7 A transparency with transmittance $f(x,y) = \exp(j\pi x^2/\lambda f)$ bends the wave at position x by an angle $\theta_x \approx -x/f$ so that it acts as a cylindrical lens with focal length f.

(4)

نسبت دامنه های عبور کرده به ورودی E_t/E_0 را ترگسیله می نامیم.

$\frac{I_t}{I_0}$ = ترگسیلیتی ←

عکس تبدیل فوریه ⇐ عکس $\frac{1}{2}$

معادله توری ⇒ $m\lambda = d\sin\theta = d\frac{Y_m}{f}$ — d دوره تناوب توری است.

↙

$$Y_m = m\left(\frac{\lambda f}{d}\right) , \quad k_Y = \frac{kY}{f} \qquad k = 2\pi/\lambda$$

بسامد فضایی در صفحه فوریه ⇒ $\nu_Y = \frac{1}{\lambda_Y} = \frac{k_Y}{2\pi} = \frac{kY}{f\,2\pi} = \frac{k}{2\pi f}\,\frac{m\lambda f}{d}$

$\nu_Y = \frac{m}{d}$ ⇒ $\nu_Y = 0$ در نقطه بسامد $m=0$ لکه مرکزی

است ← مولفه DC

$\nu_{Y_1} = 1/d$ — هم $m=1$ (مرتبه اول) در دو سمت لکه مرکزی:

لکه های بالاتر $m>1$ با مضارب صحیح فرکانس پایه ←

$\nu_{Y_m} = m\nu_{Y_1}$

در سری فوریه یک تابع مربعی:

$f(t) = \frac{1}{2} + \sum \left[\left(\frac{2}{m\pi}\right)\sin\frac{m\pi}{2}\right]\cos m\omega t$

$f(Y) = \frac{1}{2} + \sum_{m=1}^{\infty}\left[\frac{2}{m\pi}\sin\frac{m\pi}{2}\right]\cos mkY$

$$f(Y) = \frac{1}{2} + \frac{2}{\pi}\left(\cos kY + \frac{1}{3}\cos 3kY + \frac{1}{5}\cos 5kY + \dots\right)$$

جمله ثابت $k=0$ به مقدار $\frac{1}{2}$ ⇐ مولفه DC ⇐ لکه مرکزی الگوی پراش

جمله اول $k_1 = 2\pi/d$

جمله های بالاتر $3K$ و $5K$ ... سهم های بسامد بالاتر ... با افزایش a پهنای ... $F(k)$ ...

تبدیل فوریه تابع گاوسی $\sqrt{a/\pi}$

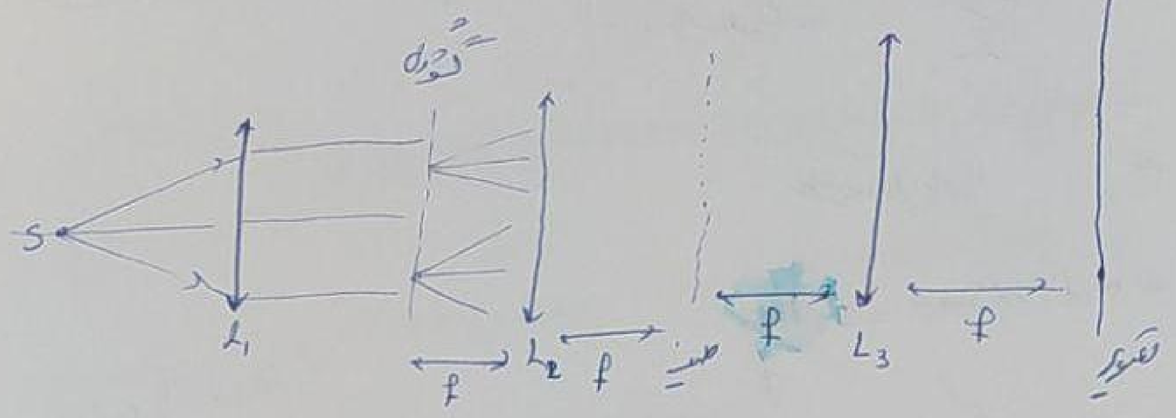
$f(x) = ce^{-ax^2}$

$F(k) = \frac{c}{\sqrt{a}}\, e^{-k^2/4a}\int_{-\infty}^{+\infty} e^{-\beta^2}\, d\beta$ → $\beta = x\sqrt{a} - \frac{ik}{2\sqrt{a}}$

$\int_{-\infty}^{+\infty} e^{-\beta^2}\, d\beta = \sqrt{\pi}$

$F(k) = e^{-k^2/4a}$ → خود یک تابع گاوسی است

پالایش اپتیکی



هر تکه پراش در صفحه طیف با مختصات (x,y) ⇐ نشانگر دسته‌های فرکانسی موجود در صفحه جسم است.

⇓

هر کدام از این تکه‌های پراش ⇐ در روشنی کل تصویر گوشه‌ای از صفحه تصویر دخالت دارند

⇓

اگر جلوی نور ناشی از یک یا چند تا از این تکه‌های پراش گرفته شود

⇓

در تصویر نهایی تغییر ایجاد می‌شود

— اگر جلوی همه‌ی نورها به جز باندهای فرکانسی بزرگتر گرفته شود ⇐ در آنجا جزئیات ظریف‌تر تصویر از بین می‌رود

— تنها DC عبور کند ⇐ صفحه تصویر روشن است. جزئیات محو می‌شود

⇐ با استفاده از یک صافی مانع شد.

— روزنه دهانه صافی رفته رفته باز شود ⇐ باندهای فرکانسی بزرگتری عبور می‌کنند ⇐ تصویر واضح‌تر می‌شود

↓

از نظر ریاضی ⇐ اضافه شدن جمله‌های با مرتبه بیشتر.

⇐ فرایند سنتز قسمت‌های مشخص (در باندهای فرکانسی مشخص) از سیستم پراش برای ساختن تصویر ⇐ تصفیه اپتیکی نام دارد.

— صافی روزنه‌ای که تمام باندها به جز باندهای نزدیک به محور مستقیم را بپوشاند ⇐ صافی اپتیکی پایین گذر است.

⇐ صافی اپتیکی بالاگذر

— تنها باندهای فرکانس [illegible]

[illegible] را بپوشاند ⇐ صافی اپتیکی نوارگذر.

همبستگی اپتیکی: ← تصویر شیء در صفحه‌ی اول با اعمال تبدیل فوریه در صفحه تصویر ← صافی اپتیکی تشکیل می‌شود.

⇓

در کانال این صفحه تصویر، شیء دوم یعنی یک عکس شفاف دیگر قرار می‌دهیم

↘

تصویر شیء اصلی روی تصویر شیء دوم می‌افتد.

↘

نور عبوری از شیء دوم در هر نقطه ← ① مقدار نور مربوط در تصویر
← ② شفافیت شیء دوم
بستگی دارد

↓

با این سیستم یک عدسی اضافه می‌کنیم و در کانون آن یک صفحه کانونی قرار می‌دهیم

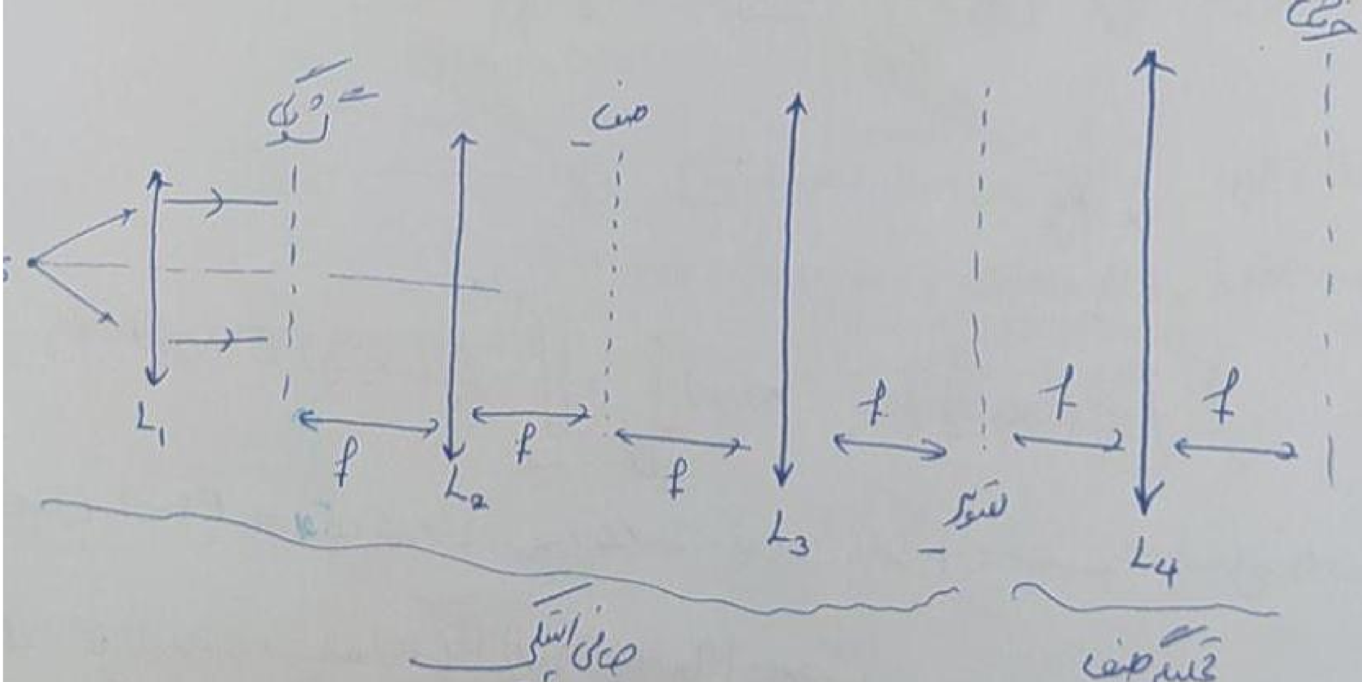
↓

تحلیلگر طیف اپتیکی

خروجی عدسی ← طیف یا تبدیل فوریه تابع تراگسیلی (انتقال عبور کرده از شیء دوم)

⇓

ابزاری برای مقایسه (همبستگی) دو شیء با توابع تراگسیلی که این دو شیء را با هم مقایسه می‌کند.



- اگر دو تابع مشابه باشند و صورتی در نظر بگیرید که تصویر اولی بر دومی منطبق باشد

⇓

عبور نور ماکسیمم است ⇐ همبستگی ماکسیمم است.

- اگر یک شیء نسبت به شیء دیگر جابجا شود ⇐ عبور نور کاهش یابد ⇐ همبستگی کاهش می‌یابد.

⇐ مانند اثر انگشت، تشخیص هویت، عکسها (ی هوایی) و ...

توصیف ریاضی:

$E_1(x,y)$ → تابع ترانسمیسیون شیء اول در صفحه کانونی که با نور تکرنگ روشن شده

$E_1(-x,-y)$ → تصویر و انتقال شیء دوم در صفحه تصویر → منفی: وارونگی تصویر نسبت به شیء

$E_2(x,y)$ → تابع ترانسمیسیون شیء دوم

نور عبور کرده: $E_1(-x,-y)\,E_2(x,y)$

تبدیل فوریه در صفحه خروجی ⇒

$$F\left[E_1(-x,-y)\,E_2(x,y)\right]=\iint_{-\infty}^{+\infty}E_1(-x,-y)\,E_2(x,y)\,e^{i(xk_x+yk_y)}\,dx$$

برای DC (مرکز سیستم) $k_x,k_y=0$

↓

$$F\left[E_1(-x,-y)\,E_2(x,y)\right]_{DC}=\iint_{-\infty}^{+\infty}E_1(-x,-y)\,E_2(x,y)\,dx\,dy$$

$E_1(x,y)$, $E_2(x,y)$ ⇐ فوتوگرافی آنها در صفحه‌های کانونی که در راستای محور اپتیکی z نسبت به هم منتقل شده‌اند

فرض: شیء اول به اندازه یک انتقال اختیاری (مولفه‌های q_x, q_y) (در صفحه شکل‌گیری) جابجا شود

Based on harmonic analysis (Fourier transform) and liner system (superposition).

An arbitrary function

$$f(x,y)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}F(\nu_x,\nu_y)\exp[-j2\pi(\nu_x x+\nu_y y)]d\nu_x d\nu_y$$

→ Superposition, or integral of harmonic functions of x and y.

$F(\nu_x,\nu_y)$: Complex amplitude

ν_x,ν_y: Spatial frequency (cycles/unit length)

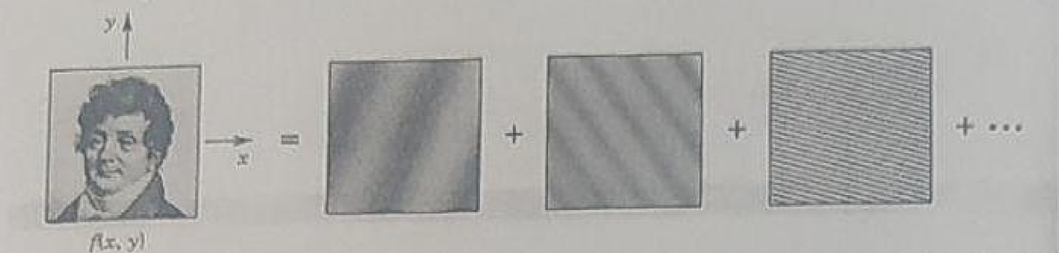


Figure 4.0-2 An arbitrary function $f(x, y)$ may be analyzed as a sum of harmonic functions of different spatial frequencies and complex amplitudes.

ompare this with plane wave

$$U(x,y,z)=A\exp[-j(k_x x+k_y y+k_z z)]$$

$$U(x,y,0)=A\exp[-j2\pi(\nu_x x+\nu_y y)]$$

$$\nu_x\leftrightarrow k_x/2\pi,\quad \nu_y\leftrightarrow k_y/2\pi$$

arbitrary function can be analyzed as a superposition of harmonic functions.
An arbitrary traveling wave $U(x,y,z)$ may be analyzed as a sum of plane ves!

opagation of Light in Free Space

respondence Between the Spatial Harmonic Function and the Plane Wave

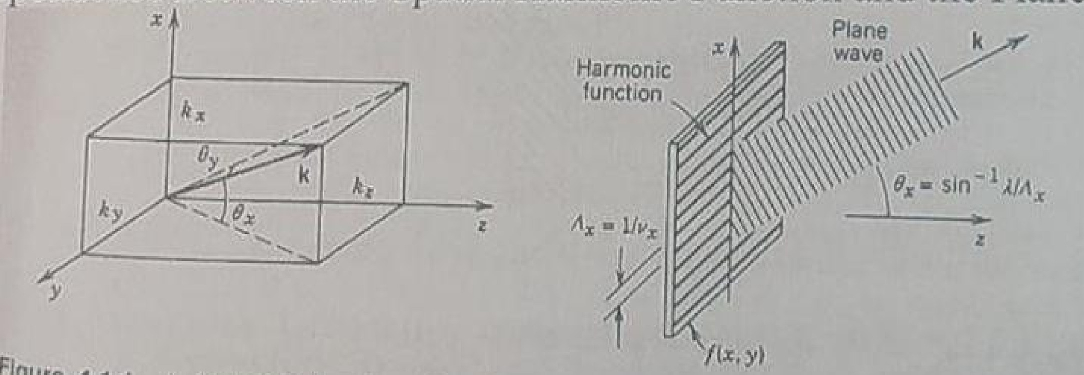


Figure 4.1-1 A harmonic function of spatial frequencies ν_x and ν_y at the plane $z = 0$ is consistent with a plane wave traveling at angles $\theta_x = \sin^{-1}\lambda\nu_x$ and $\theta_y = \sin^{-1}\lambda\nu_y$.

$$\theta_x = \sin^{-1}\left(k_x/k\right) = \sin^{-1}(\lambda\nu_x) \qquad (4.1\text{-}1)$$

$$\theta_y = \sin^{-1}\left(k_y/k\right) = \sin^{-1}(\lambda\nu_y)$$

A physical way of picturing the spatial harmonic function is to project a plane wave on the x-y plane.

$$\Lambda_x = 1/\nu_x, \quad \Lambda_y = 1/\nu_y$$

$$\rightarrow \quad \theta_x = \sin^{-1}\left(\lambda/\Lambda_x\right), \quad \theta_y = \sin^{-1}\left(\lambda/\Lambda_y\right)$$

Paraxial approximation:

$$\theta_x = \lambda/\Lambda_x = \lambda\nu_x, \quad \theta_y = \lambda/\Lambda_y = \lambda\nu_y \qquad (4.1\text{-}2)$$

Spatial spectral analysis

(Response of a plane wave after a thin optical element.)

Consider a simple case:

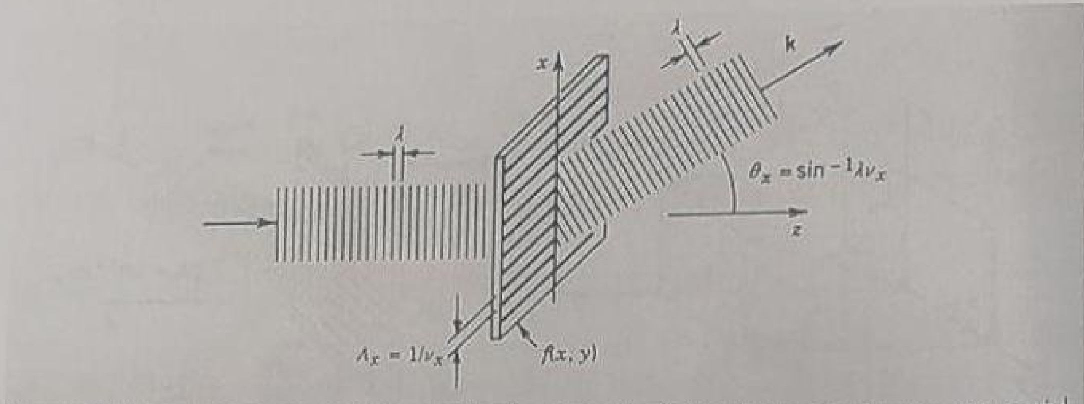


Figure 4.1-2 A thin element whose amplitude transmittance is a harmonic function of spatial frequency ν_x (period $\Lambda_x = 1/\nu_x$) bends a plane wave of wavelength λ by an angle $\theta_x = \sin^{-1}\lambda\nu_x = \sin^{-1}(\lambda/\Lambda_x)$.

$$t(x,y) = \exp[-j2\pi(\nu_x x + \nu_y y)]$$

$\rightarrow$ Harmonic function on x-y plane with period $\Lambda_x = 1/\nu_x$, $\Lambda_y = 1/\nu_y$.

$$U(x,y,z) = A\exp[-j2\pi(\nu_x x + \nu_y y)]\exp(-jk_z z)$$

$\rightarrow$ Output wave is bent with angles $\theta_x = \sin^{-1}(\lambda\nu_x)$, $\theta_y = \sin^{-1}(\lambda\nu_y)$.

The harmonic function pattern works like a grating.

w consider a general case:

$$t(x,y) = \iint F(\nu_x,\nu_y)\exp[-j2\pi(\nu_x x + \nu_y y)]d\nu_x d\nu_y \qquad (4.1\text{-}4)$$

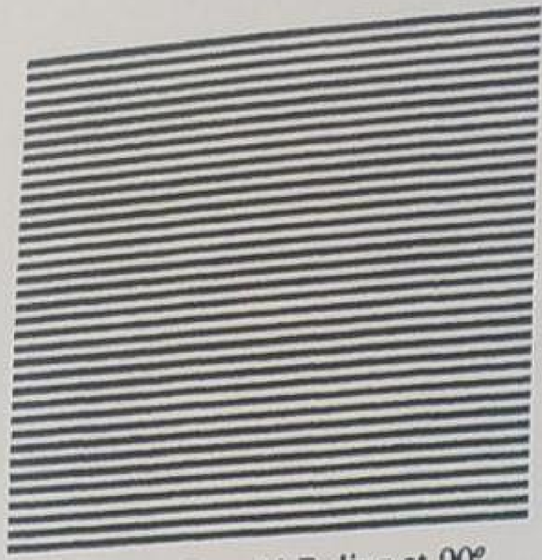


Figure 14: Ronchi Ruling at 90°

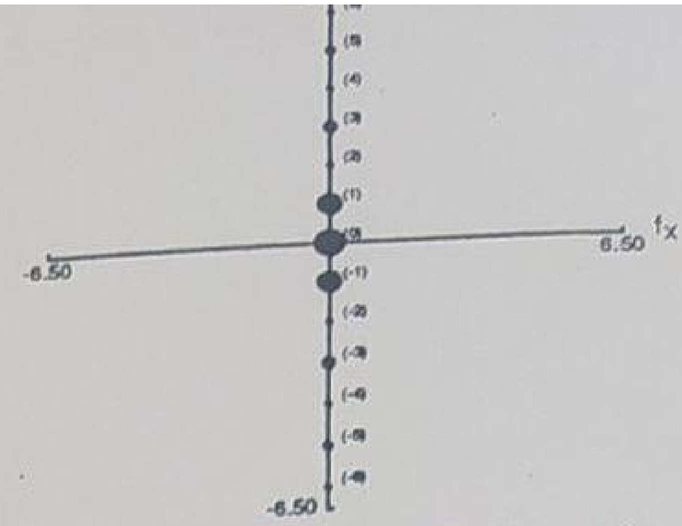


Figure 15: Fourier transform of Ronchi ruling at 90°

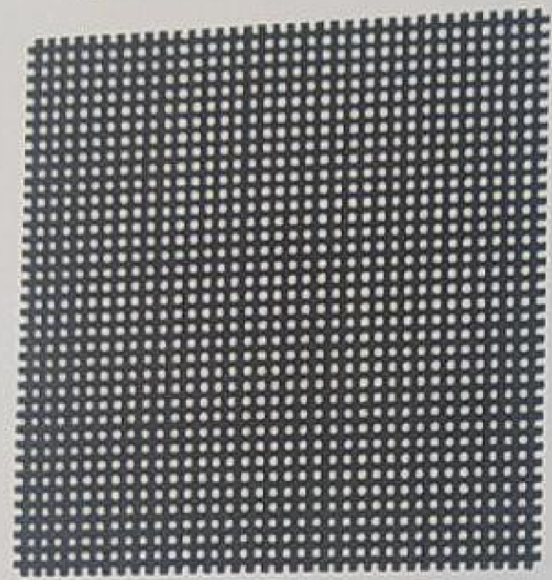


Figure 16: Crossed Ronchi rulings at 0° and 90°

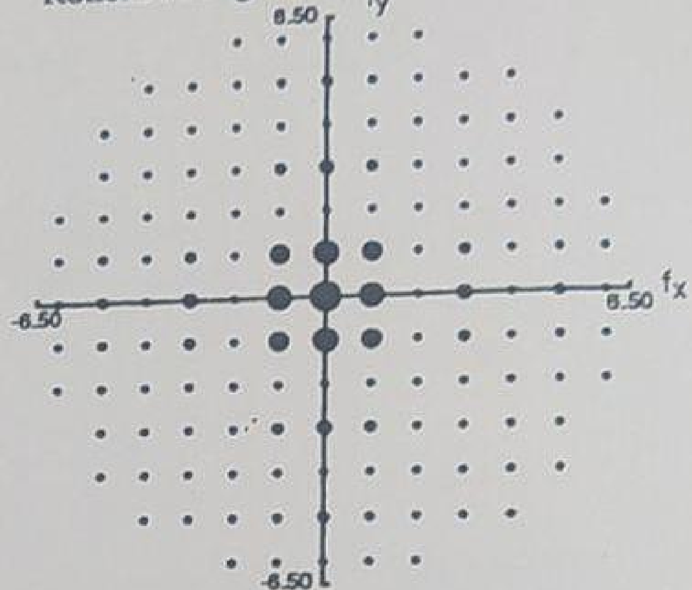


Figure 17: Fourier transform of crossed Ronchi rulings at 0° and 90°

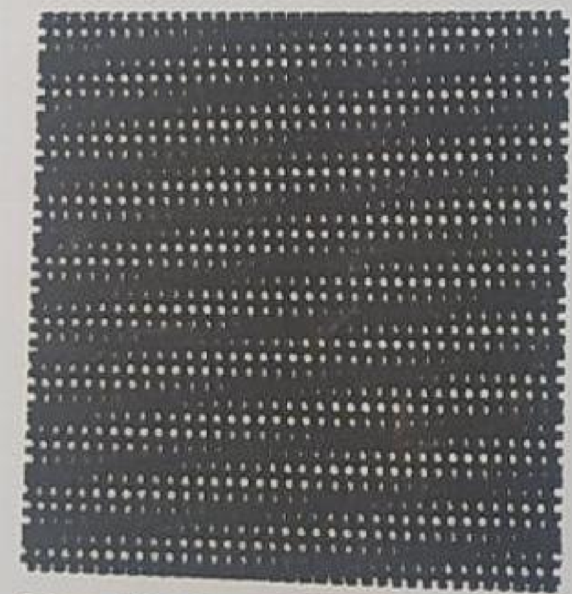


Figure 18: Crossed Ronchi rulings at 0°, 90° and 15°

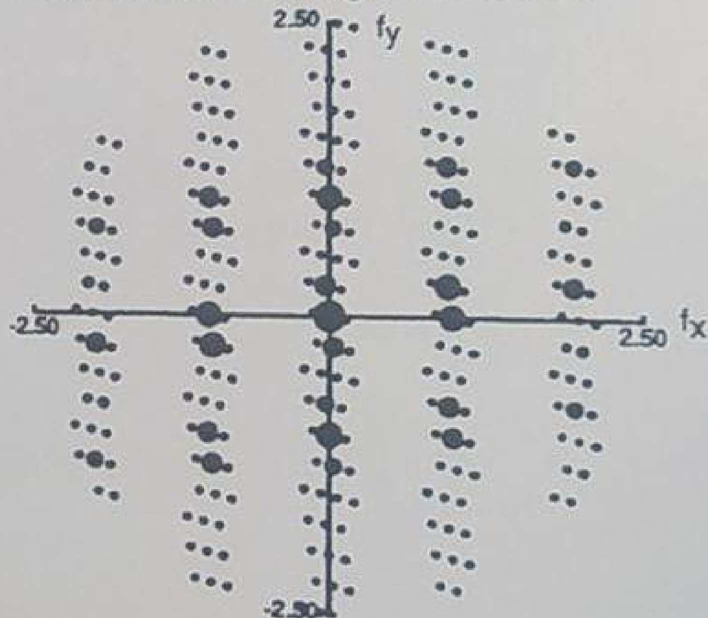


Figure 19: Fourier transform of crossed Ronchi rulings at 0°, 90° and 15°

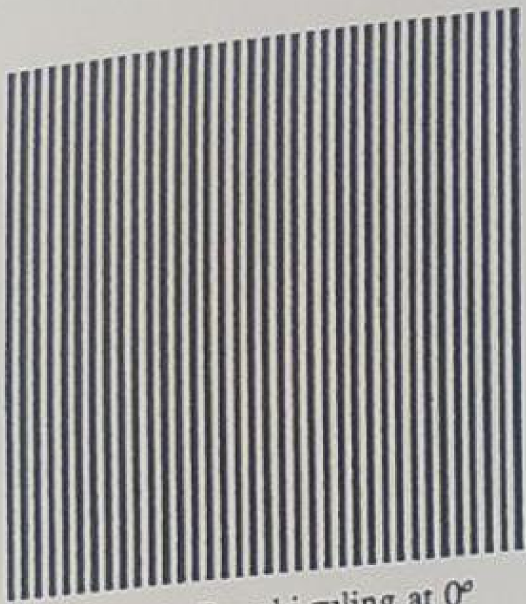


Figure 3: Ronchi ruling at 0°

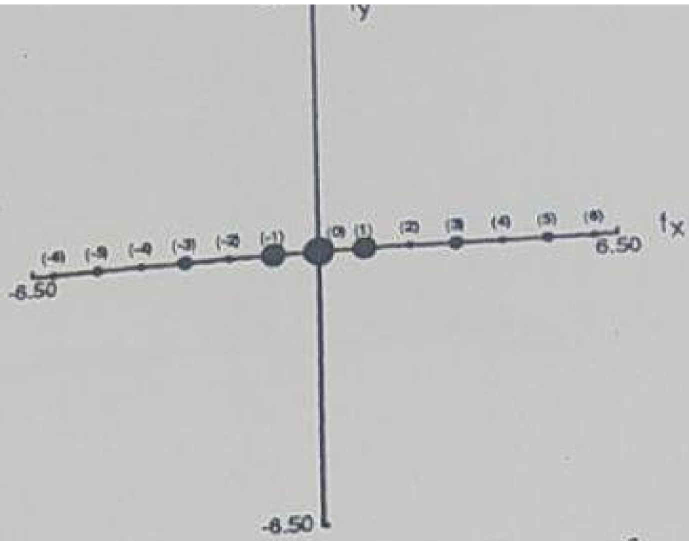


Figure 4: Fourier transform of Ronchi ruling at 0°

Figure 5: Ronchi ruling at 15°

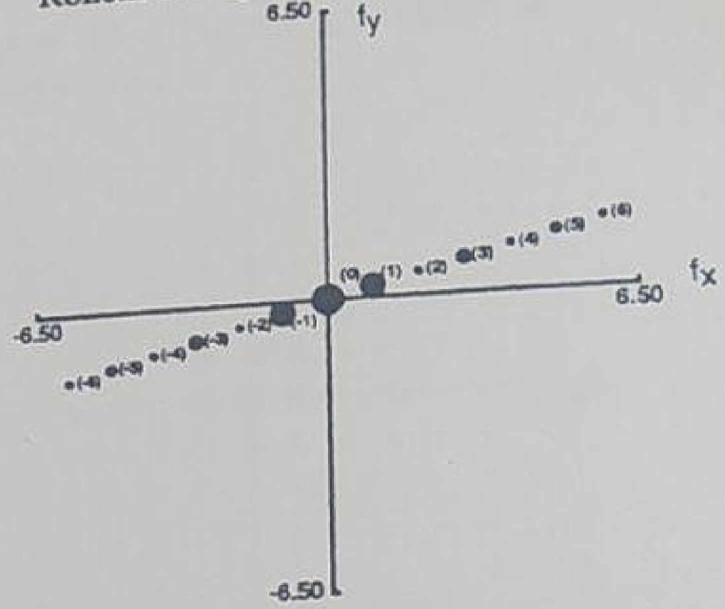


Figure 6: Fourier transform of Ronchi ruling at 15°

Figure 7: Crossed Ronchi rulings at 0° and 15°

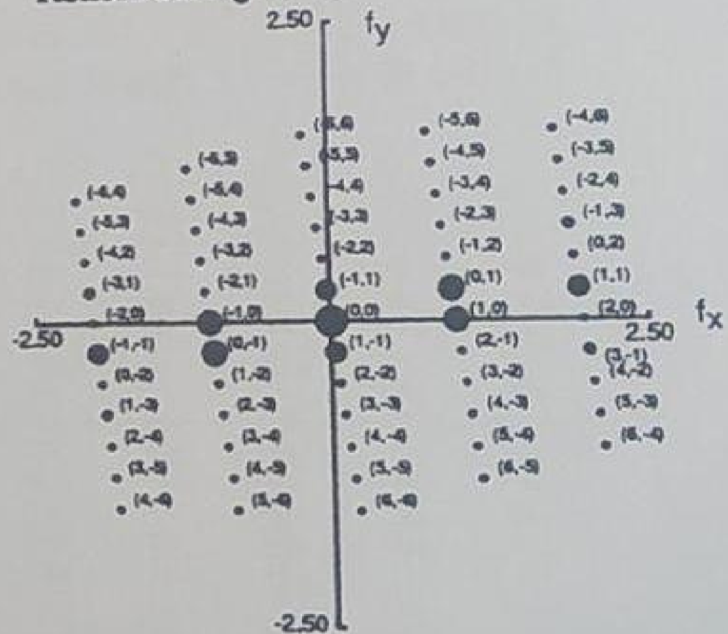


Figure 8: Fourier transform of crossed Ronchi rulings at 0° and 15°

Assume the incident wave is a plane wave of intensity I_i in z-direction.
Using Eq. (4.2-1), Fraunhofer approximation, we obtain:

$$I(x,y)=\frac{I_i}{(\lambda d)^2}\left|P(\frac{x}{\lambda d},\frac{y}{\lambda d})\right|^2 \quad (4.3\text{-}4)$$

→ Proportional to the squared magnitude of the Fourier transform of the aperture function $p(x,y)$ evaluated at the spatial frequency $\nu_x=\frac{x}{\lambda d},\nu_y=\frac{y}{\lambda d}$.

Example: Fraunhofer diffraction from a circular aperture

$$I(\rho)=\left(\frac{\pi D^2}{4\lambda d}\right)^2 I_i\left[\frac{2J_1(\pi D\rho/\lambda d)}{\pi D\rho/\lambda d}\right]^2 \quad (4.3\text{-}7)$$

→ Airy pattern. Center disk (Airy disk) has radius $\rho_s=1.22\lambda d/D$, subtending an angle $\theta=1.22\lambda/D$.

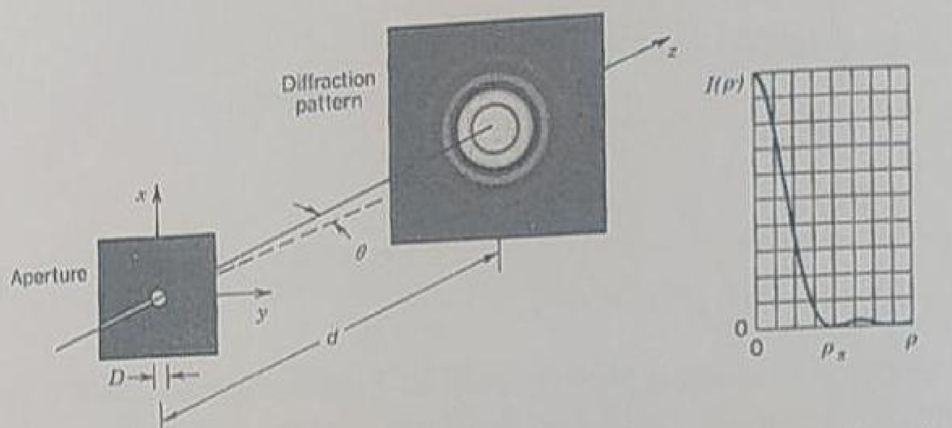


Figure 4.3-4 The Fraunhofer diffraction pattern from a circular aperture produces the Airy pattern with the radius of the central disk subtending an angle $\theta = 1.22\lambda/D$.

Fresnel Diffraction

At small distance ($d\to 0$), the diffraction pattern is the shadow of the aperture.
At medium distance (Fresnel diffraction), the diffraction pattern is the convolution of the aperture. Using Eq. (4.1-14), free-space propagation as a convolution, we obtain:

$$I(x,y)=\frac{I_i}{(\lambda d)^2}\left|\iint p(x',y')\exp\left[-j\pi\frac{(x-x')^2+(y-y')^2}{\lambda d}\right]dx'dy'\right|^2 \quad (4.3\text{-}11)$$

At large d, the diffraction pattern becomes Fraunhofer diffraction pattern. The far field has an angular divergence proportional to λ/D, where D is the diameter of the aperture.

Amplitude of the plane wave with direction $(\theta_x,\theta_y)=(\lambda\nu_x,\lambda\nu_y)$ is proportional to the Fourier transform $F(\nu_x,\nu_y)$ and is located at the point $(x,y)=(\theta_x f,\theta_y f)=(\lambda f\nu_x,\lambda f\nu_y)$.

$$\rightarrow\ g(x,y)\propto F(\frac{x}{\lambda f},\frac{y}{\lambda f}) \qquad (4.2\text{-}5)$$

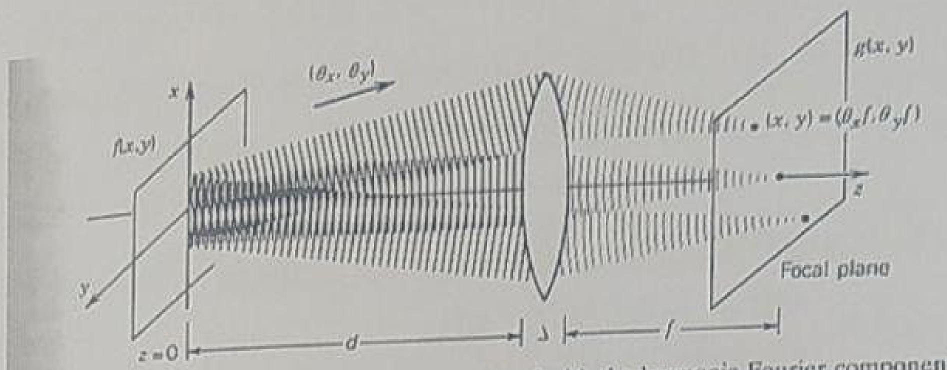


Figure 4.2-3 Focusing of the plane waves associated with the harmonic Fourier components of the input function $f(x, y)$ into points in the focal plane. The amplitude of the plane wave with direction $(\theta_x, \theta_y) = (\lambda\nu_x, \lambda\nu_y)$ is proportional to the Fourier transform $F(\nu_x, \nu_y)$ and is focused at the point $(x, y) = (\theta_x f, \theta_y f) = (\lambda f\nu_x, \lambda f\nu_y)$.

$$g(x,y)=\frac{j}{\lambda f}\exp[-jk(d+f)]\exp\left[j\pi\frac{(x^2+y^2)(d-f)}{\lambda f^2}\right]F(\frac{x}{\lambda f},\frac{y}{\lambda f}) \qquad (4.2\text{-}8)$$

$$I(x,y)=\frac{1}{|\lambda f|^2}\left|F(\frac{x}{\lambda f},\frac{y}{\lambda f})\right|^2 \qquad (4.2\text{-}9)$$

If $d=f$, $g(x,y)=\frac{j}{\lambda f}\exp[-j2kf]F(\frac{x}{\lambda f},\frac{y}{\lambda f})$ (4.2-10)

Fourier transform using a lens is valid in Fresnel approximation (only radius at the output is limited). Without the lens, we need Fraunhofer approximation (radii at both output and input are limited).

4.3 Diffraction of Light

Light not simply blocked by an opaque object, as in Ray Optics. It depends on the wavelength, the dimension of the object, and the distance between the object and the observation plane.

A. Fraunhofer Diffraction

Aperture function $p(x,y)$, with Fourier components $P(\nu_x,\nu_y)=P(\frac{x}{\lambda d},\frac{y}{\lambda d})$.

$$g(x,y) = h_0 \exp\left(-j\pi \frac{x^2+y^2}{\lambda d}\right) F\left(\frac{x}{\lambda d}, \frac{y}{\lambda d}\right) \quad (4.2\text{-}4)$$

Furthermore, if we limit our interest to points at the output plane within a circle of radius a centered about the z axis, so that $N_F = a^2/\lambda d << 1$ for $g(x,y)$.

$$g(x,y) = h_0 F\left(\frac{x}{\lambda d}, \frac{y}{\lambda d}\right) \quad (4.2\text{-}1)$$

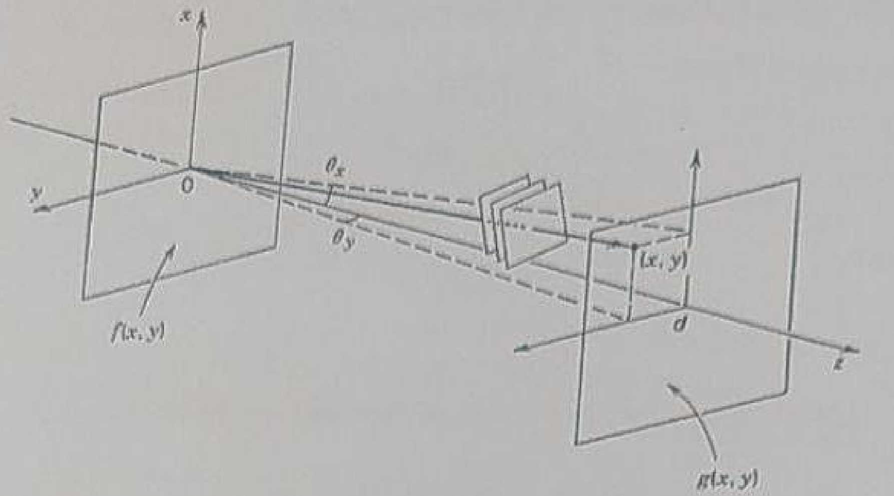


Figure 4.2-1 When the distance d is sufficiently long, the complex amplitude at point (x, y) in the $z = d$ plane is proportional to the complex amplitude of the plane-wave component with angles $\theta_x \approx x/d \approx \lambda\nu_x$ and $\theta_y \approx y/d \approx \lambda\nu_y$, i.e., to the Fourier transform $F(\nu_x, \nu_y)$ of $f(x, y)$, with $\nu_x = x/\lambda d$ and $\nu_y = y/\lambda d$.

- The only plane wave that contributes to the complex amplitude at (x, y) at tput plane is the wave making angles $\theta_x = x/d, \theta_y = y/d$ with the optical axis. is is also the wave with wave-vector components $k_x = (x/d)k, k_y = (y/d)k$ and plitude $F(\nu_x, \nu_y)$ with $\nu_x = x/\lambda d, \nu_y = y/\lambda d$.

Fraunhofer approximation is valid when both N_F and N_F' are small.

ourier Transform Using a Lens

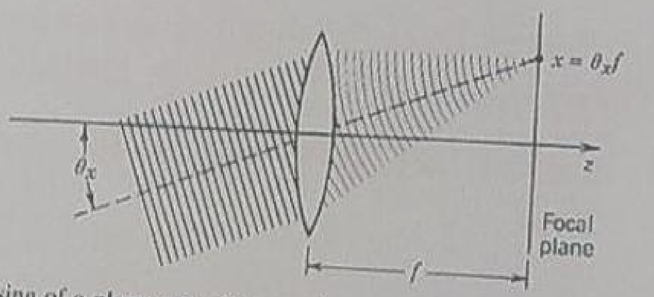


Figure 4.2-2 Focusing of a plane wave into a point. A direction (θ_x, θ_y) is mapped into a point $(x, y) = (\theta_x f, \theta_y f)$.

(2) Complex envelopes of the plane-wave components in the output plane =

$H(\nu_x,\nu_y)F(\nu_x,\nu_y)$

(3) $g(x,y)=\iint H(\nu_x,\nu_y)F(\nu_x,\nu_y)\exp[-j2\pi(\nu_x x+\nu_y y)]d\nu_x d\nu_y$

Under Fresnel approximation,

$g(x,y)=H_0\iint F(\nu_x,\nu_y)\exp[j\pi\lambda d(\nu_x^2+\nu_y^2)]\exp[-j2\pi(\nu_x x+\nu_y y)]d\nu_x d\nu_y$

$H_0\equiv\exp(-jkd)$

Free-space propagation as a convolution

Each point generates a spherical wave. Under Fresnel approximation (observation point close to the propagation axis), spherical wave → parabolic wave.

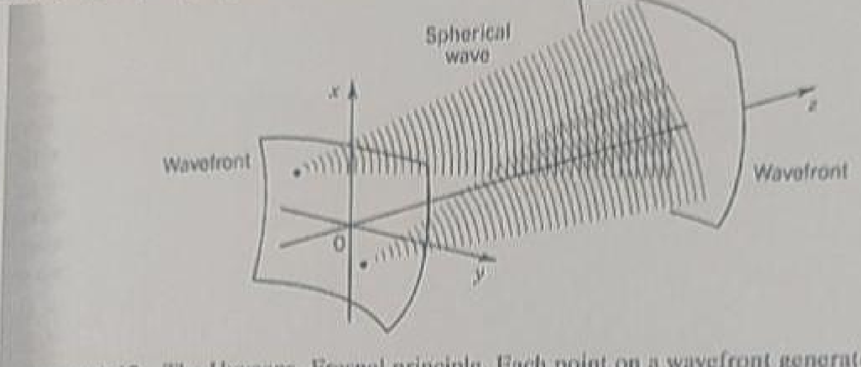


Figure 4.1-12 The Huygens–Fresnel principle. Each point on a wavefront generates a spherical wave.

$$h(x,y)\approx h_0\exp\left[-jk\frac{x^2+y^2}{2d}\right] \quad (4.1\text{-}13)$$

$$h_0=\frac{j}{\lambda d}\exp(-jkd)$$

$$g(x,y)=h_0\iint f(x',y')\exp\left[-j\pi\frac{(x-x')^2+(y-y')^2}{\lambda d}\right]dx'dy' \quad (4.1\text{-}14)$$

4.2 Optical Fourier Transform

A plane wave transmitting through an optical element can be used to decompose the harmonic functions (Fourier components $F(\nu_x,\nu_y)$) that compose the pattern ($f(x,y)$) on the optical element.

A. Fourier Transform in the Far Field (Fraunhofer Approximation)

If $f(x,y)$ is confined to a small area of radius b, distance d to the observation plane is sufficiently large, so that Fresnel number for $f(x,y)$, $N_F'=b^2/\lambda d\ll 1$.

B. Transfer Function of Free Space

Since an arbitrary function can be analyzed as sum of harmonic functions, we consider a harmonic input function.

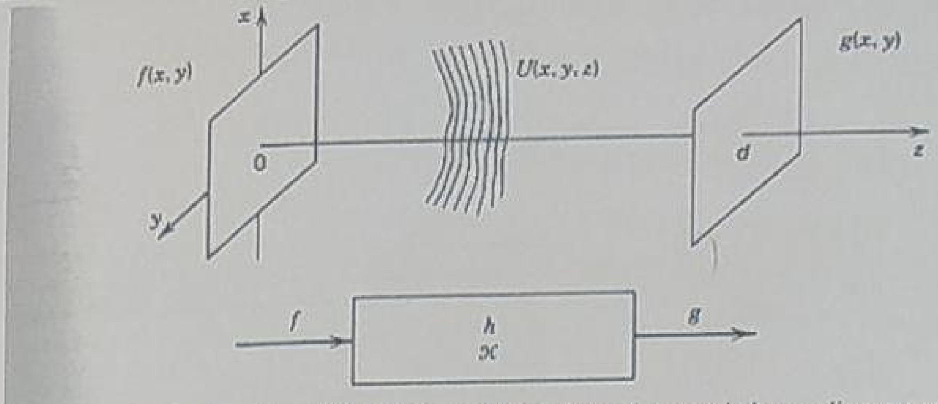


Figure 4.1-9 Propagation of light between two planes is regarded as a linear system whose input and output are the complex amplitudes of the wave in the two planes.

$$f(x,y) = U(x,y,0) = A\exp[-j2\pi(\nu_x x + \nu_y y)]$$

Output $$g(x,y) = U(x,y,d) = A\exp[-j(k_x x + k_y y + k_z d)]$$

$$\mathrm{H}(\nu_x,\nu_y) = \frac{g(x,y)}{f(x,y)} = \exp(-jk_z d)$$

$$= \exp\left[-j2\pi\left(\frac{1}{\lambda^2} - \nu_x^2 - \nu_y^2\right)^{1/2} d\right] \qquad (4.1\text{-}6)$$

Fresnel approximation

$$\nu_x^2 + \nu_y^2 << 1/\lambda^2$$

→ The plane-wave components of the propagating light make small angles $\theta_x \sim \lambda\nu_x, \theta_y \sim \lambda\nu_y$.

→ Paraxial waves:

$$H(\nu_x,\nu_y) = \exp(-jkd)\exp\left[j\pi\lambda d\left(\nu_x^2 + \nu_y^2\right)\right] \qquad (4.1\text{-}8)$$

Validity of Fresnel approximation has the same expression as in Sec. 2.2.

Input-output relation

Given the input function $f(x,y)$, how to obtain the output $g(x,y)$:

(1) Determine the complex envelopes of the plane-wave components in the input plane by Fourier transform.

$$F(\nu_x,\nu_y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\exp[j2\pi(\nu_x x + \nu_y y)]dxdy$$