In the name of God

Statics

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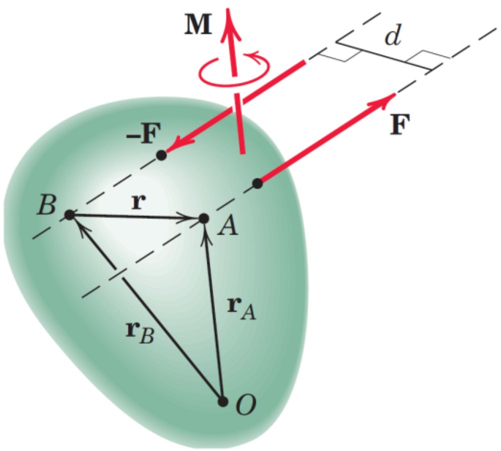
Introduction to Statics

Introduction

Force Systems

Introduction



Couples in Three Dimensions

The concept of the couple was introduced in Art. 2/5 and is easily extended to three dimensions. Figure 2/25 shows two equal and opposite forces $\mathbf{F}$ and $-\mathbf{F}$ acting on a body. The vector $\mathbf{r}$ runs from *any* point B on the line of action of $-\mathbf{F}$ to *any* point A on the line of action of $\mathbf{F}$. Points A and B are located by position vectors $\mathbf{r}_A$ and $\mathbf{r}_B$ from *any* point O. The combined moment of the two forces about O is

$$\mathbf{M} = \mathbf{r}_A \times \mathbf{F} + \mathbf{r}_B \times (-\mathbf{F}) = (\mathbf{r}_A - \mathbf{r}_B) \times \mathbf{F}$$

However, $\mathbf{r}_A - \mathbf{r}_B = \mathbf{r}$, so that all reference to the moment center O disappears, and the moment of the couple becomes

$$\mathbf{M} = \mathbf{r} \times \mathbf{F} \qquad \mathbf{(2/19)}$$

Thus, the moment of a couple is the *same about all points*. The magnitude of $\mathbf{M}$ is $M = Fd$, where d is the perpendicular distance between the lines of action of the two forces, as described in Art. 2/5.

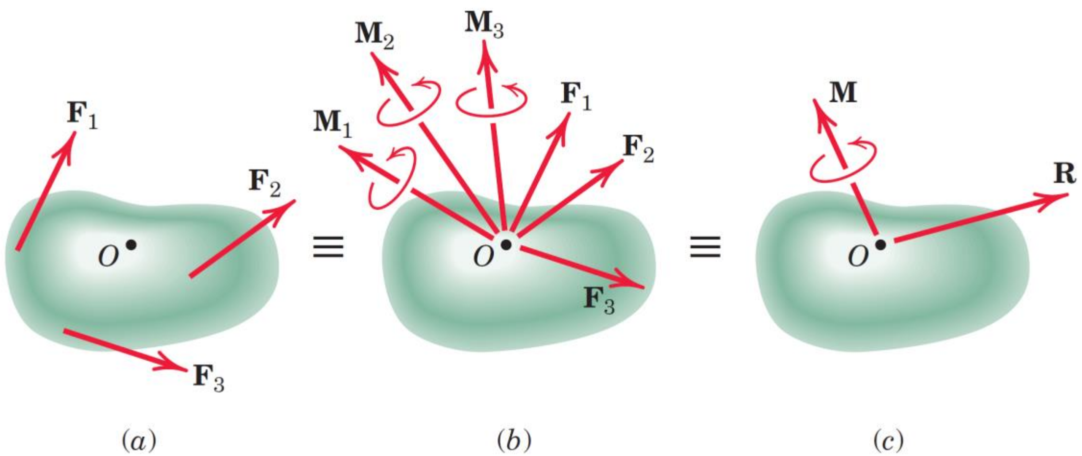
The moment of a couple is a *free vector*, whereas the moment of a force about a point (which is also the moment about a defined axis through the point) is a *sliding vector* whose direction is along the axis through the point. As in the case of two dimensions, a couple tends to produce a pure rotation of the body about an axis normal to the plane of the forces which constitute the couple.

Resultants

$$\mathbf{R} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 + \cdots = \Sigma\mathbf{F}$$
$$\mathbf{M} = \mathbf{M}_1 + \mathbf{M}_2 + \mathbf{M}_3 + \cdots = \Sigma(\mathbf{r} \times \mathbf{F}) \tag{2/20}$$

The couple vectors are shown through point O, but because they are free vectors, they may be represented in any parallel positions. The magnitudes of the resultants and their components are

$$R_x = \Sigma F_x \qquad R_y = \Sigma F_y \qquad R_z = \Sigma F_z$$
$$R = \sqrt{(\Sigma F_x)^2 + (\Sigma F_y)^2 + (\Sigma F_z)^2}$$
$$\mathbf{M}_x = \Sigma(\mathbf{r} \times \mathbf{F})_x \qquad \mathbf{M}_y = \Sigma(\mathbf{r} \times \mathbf{F})_y \qquad \mathbf{M}_z = \Sigma(\mathbf{r} \times \mathbf{F})_z \tag{2/21}$$
$$M = \sqrt{M_x^2 + M_y^2 + M_z^2}$$



SAMPLE PROBLEM

Determine the resultant of the force and couple system which acts on the rectangular solid.

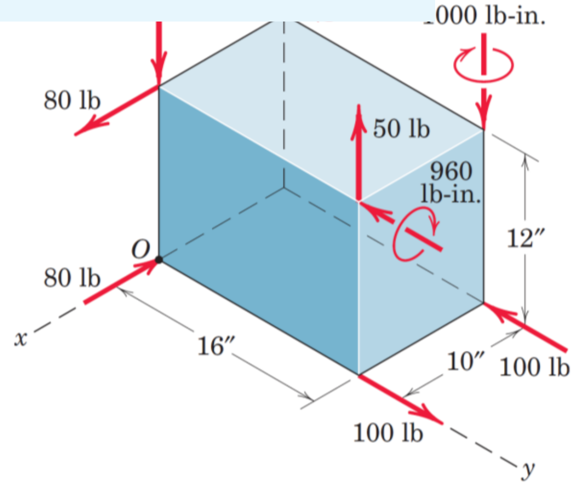
Solution. We choose point O as a convenient reference point for the initial step of reducing the given forces to a force–couple system. The resultant force is

1
$$\mathbf{R} = \Sigma\mathbf{F} = (80 - 80)\mathbf{i} + (100 - 100)\mathbf{j} + (50 - 50)\mathbf{k} = \mathbf{0} \text{ lb}$$

The sum of the moments about O is

2
$$\begin{aligned}\mathbf{M}_O &= [50(16) - 700]\mathbf{i} + [80(12) - 960]\mathbf{j} + [100(10) - 1000]\mathbf{k} \text{ lb-in.} \\ &= 100\mathbf{i} \text{ lb-in.}\end{aligned}$$

Hence, the resultant consists of a couple, which of course may be applied at any point on the body or the body extended.



Equilibrium

Introduction

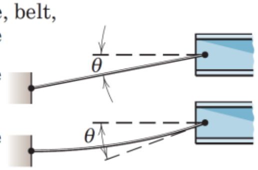
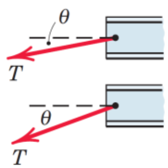
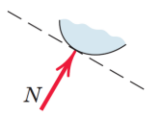
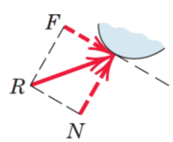
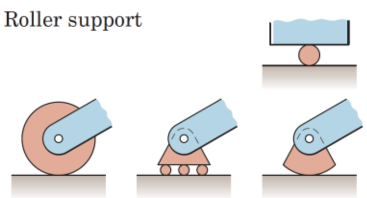
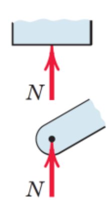
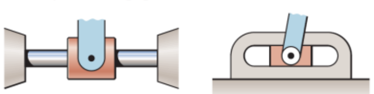
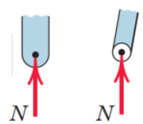
Statics deals primarily with the description of the force conditions necessary and sufficient to maintain the equilibrium of engineering structures. This chapter on equilibrium, therefore, constitutes the most important part of statics, and the procedures developed here form the basis for solving problems in both statics and dynamics. We will make continual use of the concepts developed in Chapter 2 involving forces, moments, couples, and resultants as we apply the principles of equilibrium.

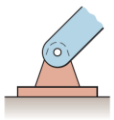
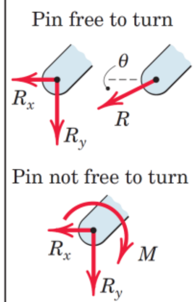
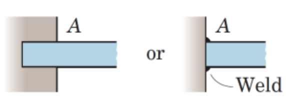
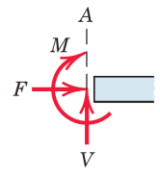
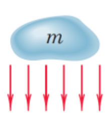
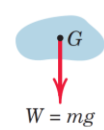
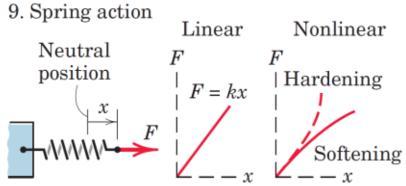
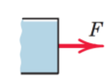
When a body is in equilibrium, the resultant of *all* forces acting on it is zero. Thus, the resultant force **R** and the resultant couple **M** are both zero, and we have the equilibrium equations

$$\mathbf{R} = \Sigma\mathbf{F} = \mathbf{0} \qquad \mathbf{M} = \Sigma\mathbf{M} = \mathbf{0} \tag{3/1}$$

Equilibrium

System Isolation and the Free-Body Diagram

MODELING THE ACTION OF FORCES IN TWO-DIMENSIONAL ANALYSIS	
Type of Contact and Force Origin	**Action on Body to Be Isolated**
1. Flexible cable, belt, chain, or rope Weight of cable negligible Weight of cable not negligible 	 Force exerted by a flexible cable is always a tension away from the body in the direction of the cable.
2. Smooth surfaces 	 Contact force is compressive and is normal to the surface.
3. Rough surfaces 	 Rough surfaces are capable of supporting a tangential component F (frictional force) as well as a normal component N of the resultant contact force R.
4. Roller support 	 Roller, rocker, or ball support transmits a compressive force normal to the supporting surface.
5. Freely sliding guide 	 Collar or slider free to move along smooth guides; can support force normal to guide only.

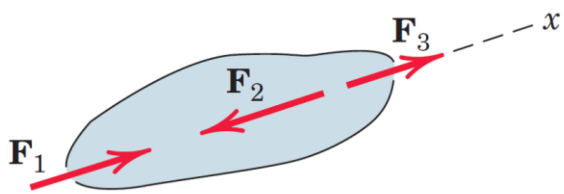
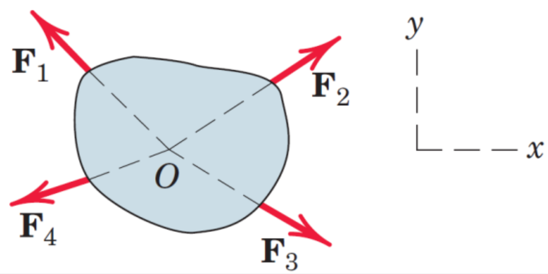
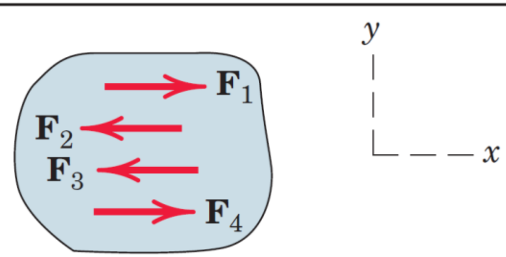
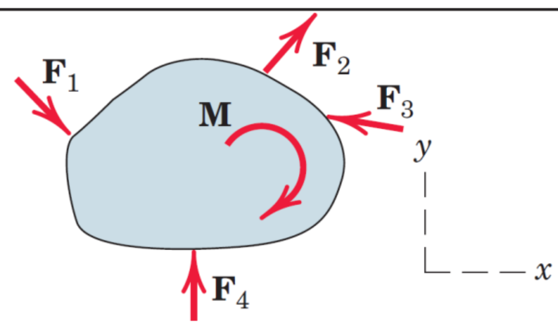
MODELING THE ACTION OF FORCES IN TWO-DIMENSIONAL ANALYSIS (*cont.*)	
Type of Contact and Force Origin	**Action on Body to Be Isolated**
6. Pin connection 	 A freely hinged pin connection is capable of supporting a force in any direction in the plane normal to the pin axis. We may either show two components R_x and R_y or a magnitude R and direction θ. A pin not free to turn also supports a couple M.
7. Built-in or fixed support 	 A built-in or fixed support is capable of supporting an axial force F, a transverse force V (shear force), and a couple M (bending moment) to prevent rotation.
8. Gravitational attraction 	 The resultant of gravitational attraction on all elements of a body of mass m is the weight $W = mg$ and acts toward the center of the earth through the center mass G.
9. Spring action 	 Spring force is tensile if spring is stretched and compressive if compressed. For a linearly elastic spring the stiffness k is the force required to deform the spring a unit distance.

Equilibrium Conditions

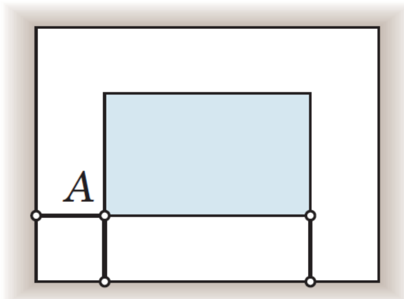
In Art. 3/1 we defined equilibrium as the condition in which the resultant of all forces and moments acting on a body is zero. Stated in another way, a body is in equilibrium if all forces and moments applied to it are in balance. These requirements are contained in the vector equations of equilibrium, Eqs. 3/1, which in two dimensions may be written in scalar form as

$$\Sigma F_x = 0 \qquad \Sigma F_y = 0 \qquad \Sigma M_O = 0 \tag{3/2}$$

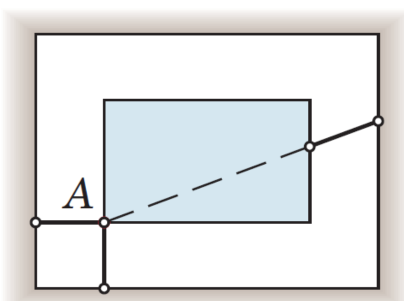
Equilibrium

<table>
<tr><th colspan="3">CATEGORIES OF EQUILIBRIUM IN TWO DIMENSIONS</th></tr>
<tr><th>Force System</th><th>Free-Body Diagram</th><th>Independent Equations</th></tr>
<tr><td>1. Collinear</td><td>
</td><td>$\Sigma F_x = 0$</td></tr>
<tr><td>2. Concurrent at a point</td><td>
</td><td>$\Sigma F_x = 0$
$\Sigma F_y = 0$</td></tr>
<tr><td>3. Parallel</td><td>
</td><td>$\Sigma F_x = 0 \quad \Sigma M_z = 0$</td></tr>
<tr><td>4. General</td><td>
</td><td>$\Sigma F_x = 0 \quad \Sigma M_z = 0$
$\Sigma F_y = 0$</td></tr>
</table>

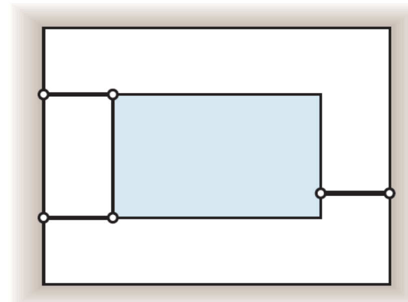
Constraints and Statical Determinacy



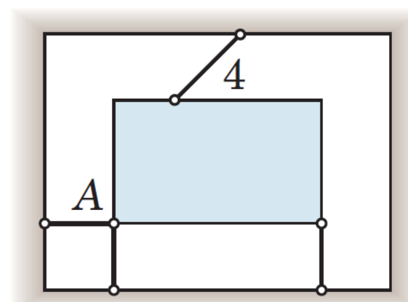
(a) Complete fixity
Adequate constraints



(b) Incomplete fixity
Partial constraints



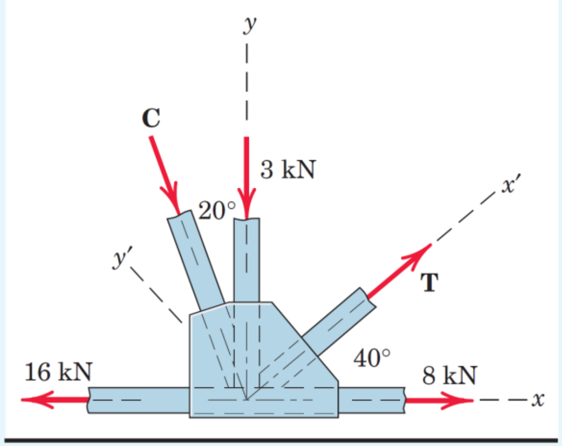
(c) Incomplete fixity
Partial constraints



(d) Excessive fixity
Redundant constraint

SAMPLE PROBLEM

Determine the magnitudes of the forces **C** and **T**, which, along with the other three forces shown, act on the bridge-truss joint.



Solution. The given sketch constitutes the free-body diagram of the isolated section of the joint in question and shows the five forces which are in equilibrium. 1

Solution 1 (scalar algebra). For the x-y axes as shown we have

$$[\Sigma F_x = 0] \qquad 8 + T\cos 40° + C\sin 20° - 16 = 0$$

$$0.766T + 0.342C = 8 \qquad (a)$$

$$[\Sigma F_y = 0] \qquad T\sin 40° - C\cos 20° - 3 = 0$$

$$0.643T - 0.940C = 3 \qquad (b)$$

Simultaneous solution of Eqs. (a) and (b) produces

$$T = 9.09 \text{ kN} \qquad C = 3.03 \text{ kN} \qquad \textit{Ans.}$$

Solution II (scalar algebra). To avoid a simultaneous solution, we may use axes x'-y' with the first summation in the y'-direction to eliminate reference to T. Thus, 2

$$[\Sigma F_{y'} = 0] \qquad -C\cos 20° - 3\cos 40° - 8\sin 40° + 16\sin 40° = 0$$

$$C = 3.03 \text{ kN} \qquad \textit{Ans.}$$

$$[\Sigma F_{x'} = 0] \qquad T + 8\cos 40° - 16\cos 40° - 3\sin 40° - 3.03\sin 20° = 0$$

$$T = 9.09 \text{ kN} \qquad \textit{Ans.}$$

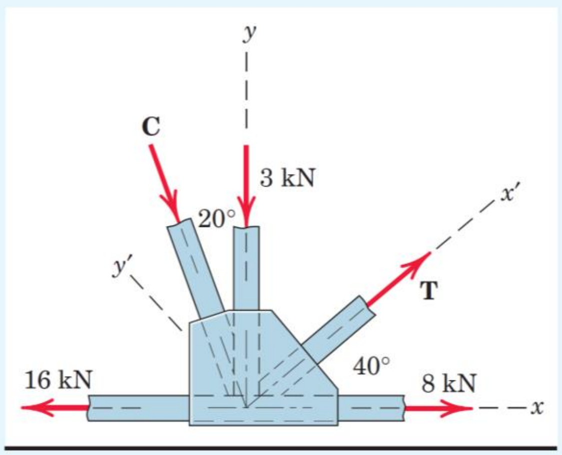
Solution III (vector algebra). With unit vectors **i** and **j** in the x- and y-directions, the zero summation of forces for equilibrium yields the vector equation

$$[\Sigma \mathbf{F} = \mathbf{0}] \qquad 8\mathbf{i} + (T\cos 40°)\mathbf{i} + (T\sin 40°)\mathbf{j} - 3\mathbf{j} + (C\sin 20°)\mathbf{i} - (C\cos 20°)\mathbf{j} - 16\mathbf{i} = \mathbf{0}$$

Equating the coefficients of the **i**- and **j**-terms to zero gives

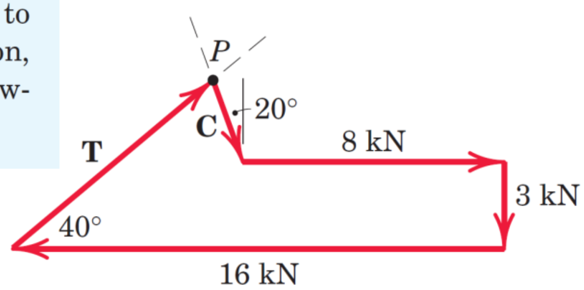
$$8 + T\cos 40° + C\sin 20° - 16 = 0$$
$$T\sin 40° - 3 - C\cos 20° = 0$$

which are the same, of course, as Eqs. (a) and (b), which we solved above.



Solution IV (geometric). The polygon representing the zero vector sum of the five forces is shown. Equations (a) and (b) are seen immediately to give the projections of the vectors onto the x- and y-directions. Similarly, projections onto the x'- and y'-directions give the alternative equations in Solution II.

A graphical solution is easily obtained. The known vectors are laid off head-to-tail to some convenient scale, and the directions of **T** and **C** are then drawn to close the polygon. The resulting intersection at point P completes the solution, thus enabling us to measure the magnitudes of **T** and **C** directly from the drawing to whatever degree of accuracy we incorporate in the construction.



SECTION B EQUILIBRIUM IN THREE DIMENSIONS

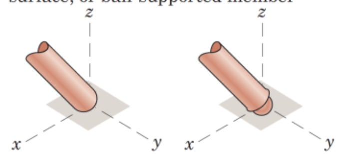
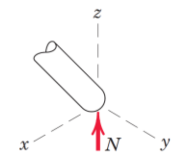
Equilibrium Conditions

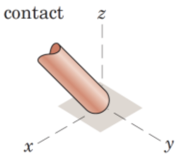
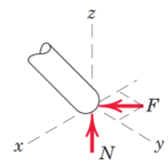
We now extend our principles and methods developed for two-dimensional equilibrium to the case of three-dimensional equilibrium. In Art. 3/1 the general conditions for the equilibrium of a body were stated in Eqs. 3/1, which require that the resultant force and resultant couple on a body in equilibrium be zero. These two vector equations of equilibrium and their scalar components may be written as

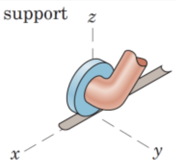
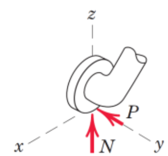
$$\Sigma \mathbf{F} = \mathbf{0} \quad \text{or} \quad \begin{cases} \Sigma F_x = 0 \\ \Sigma F_y = 0 \\ \Sigma F_z = 0 \end{cases}$$

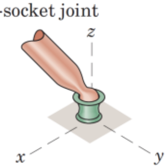
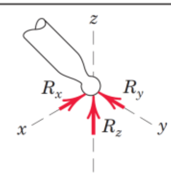
$$\Sigma \mathbf{M} = \mathbf{0} \quad \text{or} \quad \begin{cases} \Sigma M_x = 0 \\ \Sigma M_y = 0 \\ \Sigma M_z = 0 \end{cases} \qquad \textbf{(3/3)}$$

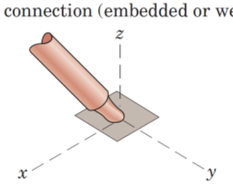
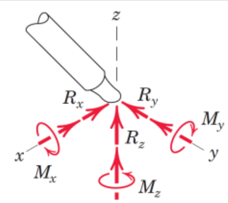
Equilibrium

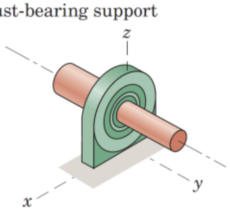
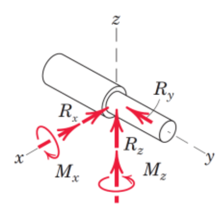
<table>
<tr><th colspan="2">MODELING THE ACTION OF FORCES IN THREE-DIMENSIONAL ANALYSIS</th></tr>
<tr><th>Type of Contact and Force Origin</th><th>Action on Body to Be Isolated</th></tr>
<tr><td>1. Member in contact with smooth surface, or ball-supported member
</td><td>

Force must be normal to the surface and directed toward the member.</td></tr>
<tr><td>2. Member in contact with rough surface
</td><td>

The possibility exists for a force F tangent to the surface (friction force) to act on the member, as well as a normal force N.</td></tr>
<tr><td>3. Roller or wheel support with lateral constraint
</td><td>

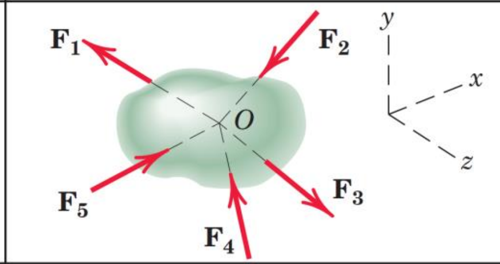
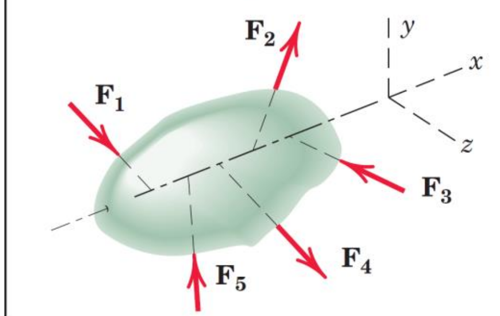
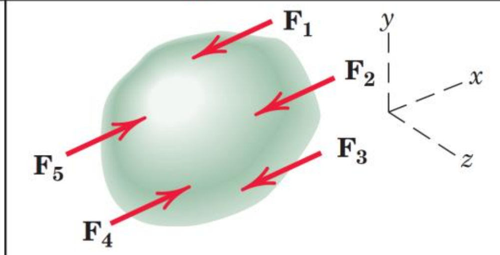
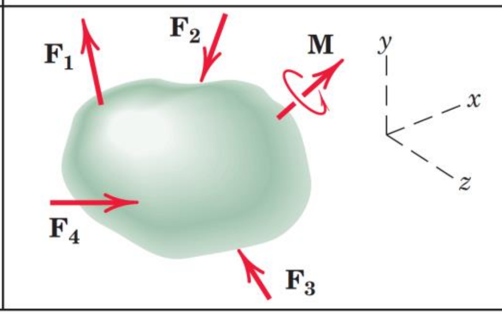
A lateral force P exerted by the guide on the wheel can exist, in addition to the normal force N.</td></tr>
<tr><td>4. Ball-and-socket joint
</td><td>

A ball-and-socket joint free to pivot about the center of the ball can support a force R with all three components.</td></tr>
<tr><td>5. Fixed connection (embedded or welded)
</td><td>

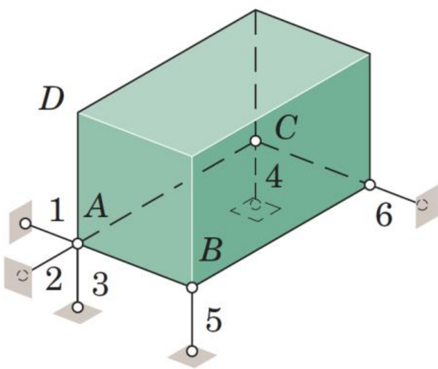
In addition to three components of force, a fixed connection can support a couple M represented by its three components.</td></tr>
<tr><td>6. Thrust-bearing support
</td><td>

Thrust bearing is capable of supporting axial force R_y as well as radial forces R_x and R_z. Couples M_x and M_z must, in some cases, be assumed zero in order to provide statical determinacy.</td></tr>
</table>

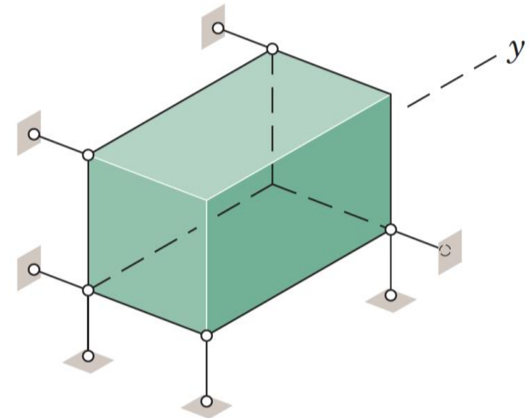
Equilibrium

CATEGORIES OF EQUILIBRIUM IN THREE DIMENSIONS		
Force System	Free-Body Diagram	Independent Equations
1. Concurrent at a point		$\Sigma F_x = 0$ $\Sigma F_y = 0$ $\Sigma F_z = 0$
2. Concurrent with a line		$\Sigma F_x = 0$ $\Sigma M_y = 0$ $\Sigma F_y = 0$ $\Sigma M_z = 0$ $\Sigma F_z = 0$
3. Parallel		$\Sigma F_x = 0$ $\Sigma M_y = 0$ $\Sigma M_z = 0$
4. General		$\Sigma F_x = 0$ $\Sigma M_x = 0$ $\Sigma F_y = 0$ $\Sigma M_y = 0$ $\Sigma F_z = 0$ $\Sigma M_z = 0$

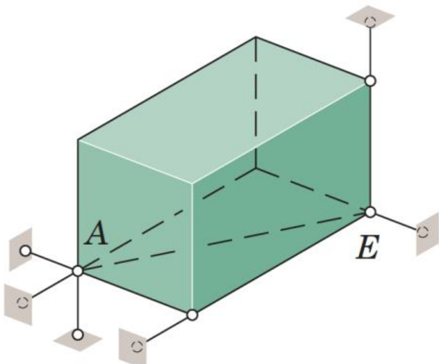
Constraints and Statical Determinacy



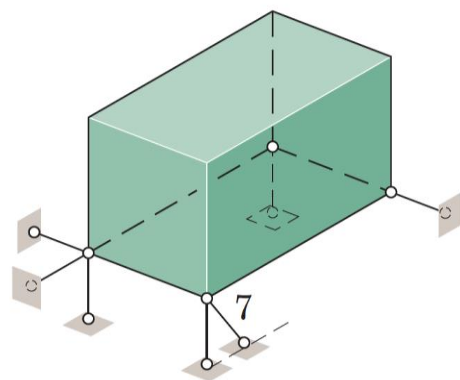
(*a*) Complete fixity
Adequate constraints



(*c*) Incomplete fixity
Partial constraints



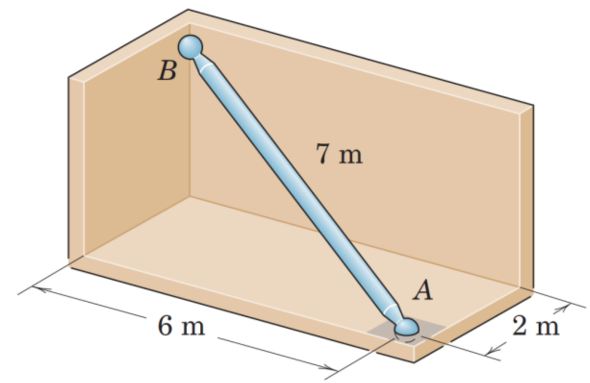
(*b*) Incomplete fixity
Partial constraints



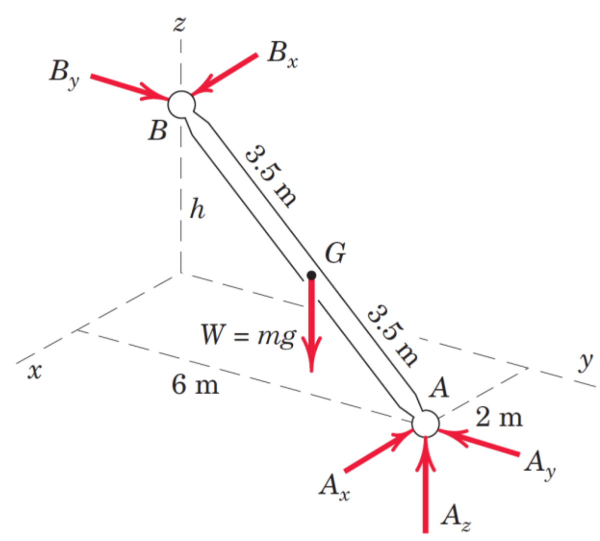
(*d*) Excessive fixity
Redundant constraints

SAMPLE PROBLEM

The uniform 7-m steel shaft has a mass of 200 kg and is supported by a ball-and-socket joint at A in the horizontal floor. The ball end B rests against the smooth vertical walls as shown. Compute the forces exerted by the walls and the floor on the ends of the shaft.



Solution. The free-body diagram of the shaft is first drawn where the contact forces acting on the shaft at B are shown normal to the wall surfaces. In addition to the weight $W = mg = 200(9.81) = 1962$ N, the force exerted by the floor on the ball joint at A is represented by its x-, y-, and z-components. These components are shown in their correct physical sense, as should be evident from the requirement that A be held in place. The vertical position of B is found from $7 = \sqrt{2^2 + 6^2 + h^2}$, $h = 3$ m. Right-handed coordinate axes are assigned as shown.



Vector solution. We will use A as a moment center to eliminate reference to the forces at A. The position vectors needed to compute the moments about A are

$$\mathbf{r}_{AG} = -1\mathbf{i} - 3\mathbf{j} + 1.5\mathbf{k}\ \text{m} \qquad \text{and} \qquad \mathbf{r}_{AB} = -2\mathbf{i} - 6\mathbf{j} + 3\mathbf{k}\ \text{m}$$

where the mass center G is located halfway between A and B.

The vector moment equation gives

$$[\Sigma \mathbf{M}_A = \mathbf{0}] \qquad \mathbf{r}_{AB} \times (\mathbf{B}_x + \mathbf{B}_y) + \mathbf{r}_{AG} \times \mathbf{W} = \mathbf{0}$$

$$(-2\mathbf{i} - 6\mathbf{j} + 3\mathbf{k}) \times (B_x\mathbf{i} + B_y\mathbf{j}) + (-\mathbf{i} - 3\mathbf{j} + 1.5\mathbf{k}) \times (-1962\mathbf{k}) = \mathbf{0}$$

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & -6 & 3 \\ B_x & B_y & 0 \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & -3 & 1.5 \\ 0 & 0 & -1962 \end{vmatrix} = \mathbf{0}$$

$$(-3B_y + 5890)\mathbf{i} + (3B_x - 1962)\mathbf{j} + (-2B_y + 6B_x)\mathbf{k} = \mathbf{0}$$

Equating the coefficients of $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ to zero and solving give

2 $$B_x = 654\ \text{N} \qquad \text{and} \qquad B_y = 1962\ \text{N} \qquad \textit{Ans.}$$

The forces at A are easily determined by

$$[\Sigma \mathbf{F} = \mathbf{0}] \qquad (654 - A_x)\mathbf{i} + (1962 - A_y)\mathbf{j} + (-1962 + A_z)\mathbf{k} = \mathbf{0}$$

and $\qquad A_x = 654$ N $\qquad A_y = 1962$ N $\qquad A_z = 1962$ N

Finally,

$$A = \sqrt{A_x^2 + A_y^2 + A_z^2}$$

$$= \sqrt{(654)^2 + (1962)^2 + (1962)^2} = 2850\ \text{N} \qquad \textit{Ans.}$$

Scalar solution. Evaluating the scalar moment equations about axes through A parallel, respectively, to the x- and y-axes, gives

$[\Sigma M_{A_x} = 0]$ $\quad 1962(3) - 3B_y = 0 \quad B_y = 1962 \text{ N}$

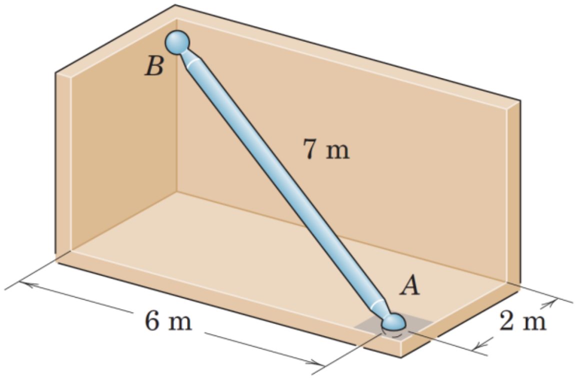
3 $[\Sigma M_{A_y} = 0]$ $\quad -1962(1) + 3B_x = 0 \quad B_x = 654 \text{ N}$

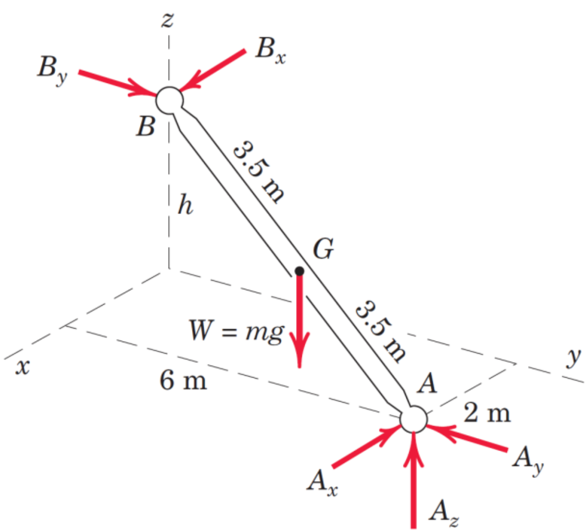
The force equations give, simply,

$[\Sigma F_x = 0]$ $\quad -A_x + 654 = 0 \quad A_x = 654 \text{ N}$

$[\Sigma F_y = 0]$ $\quad -A_y + 1962 = 0 \quad A_y = 1962 \text{ N}$

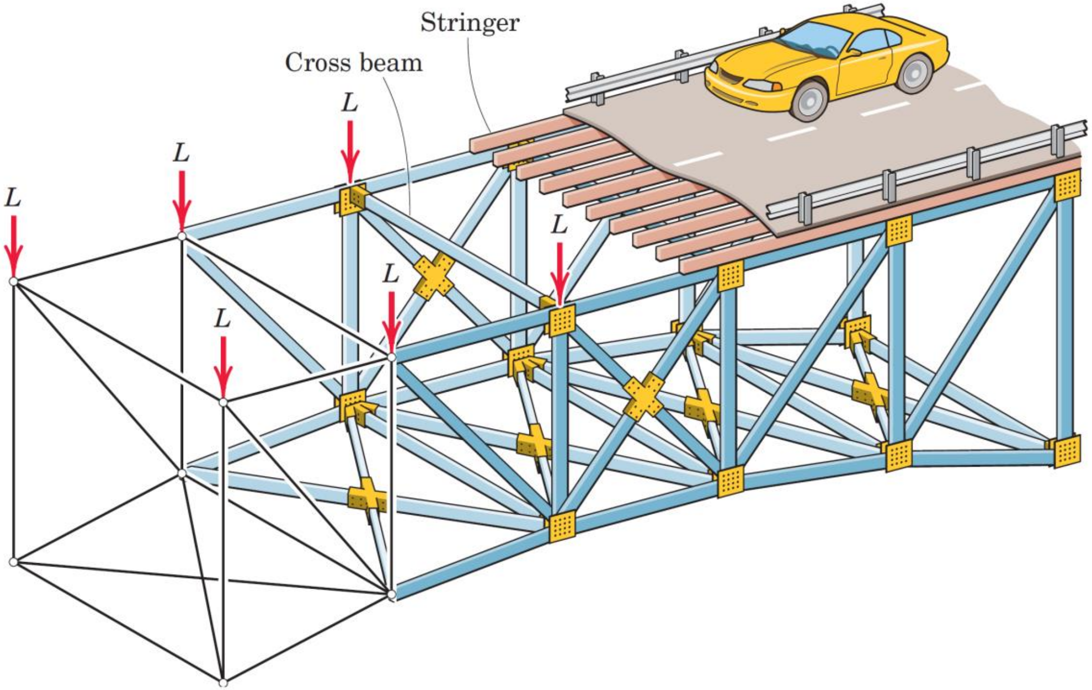
$[\Sigma F_z = 0]$ $\quad A_z - 1962 = 0 \quad A_z = 1962 \text{ N}$



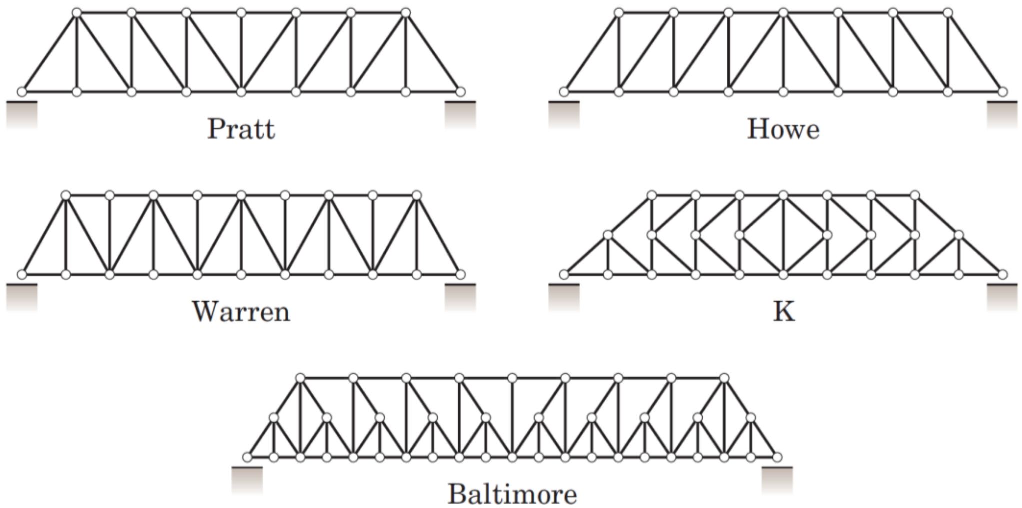


Structures

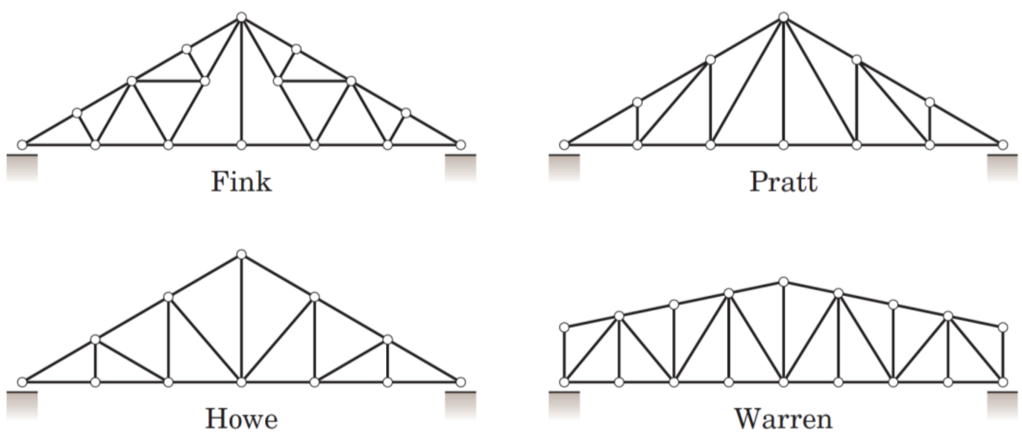
Introduction



Structures



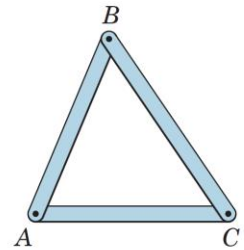
Commonly Used Bridge Trusses



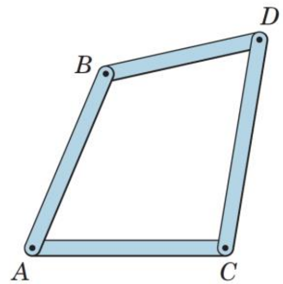
Several examples of commonly used trusses

Simple Trusses

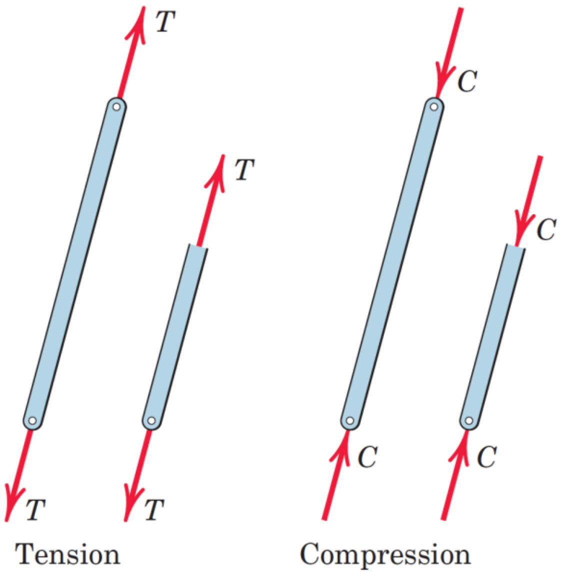
The basic element of a plane truss is the triangle. Three bars joined by pins at their ends, Fig. 4/3*a*, constitute a rigid frame. The term *rigid* is used to mean noncollapsible and also to mean that deformation of the members due to induced internal strains is negligible. On the other hand, four or more bars pin-jointed to form a polygon of as many sides constitute a nonrigid frame. We can make the nonrigid frame in Fig. 4/3*b* rigid, or stable, by adding a diagonal bar joining *A* and *D* or *B* and *C* and thereby forming two triangles. We can extend the structure by adding additional units of two end-connected bars, such as *DE* and *CE* or *AF* and *DF*, Fig. 4/3*c*, which are pinned to two fixed joints. In this way the entire structure will remain rigid.



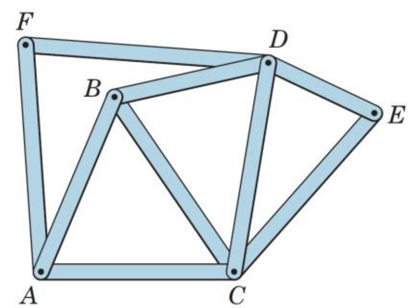
(*a*)



(*b*)

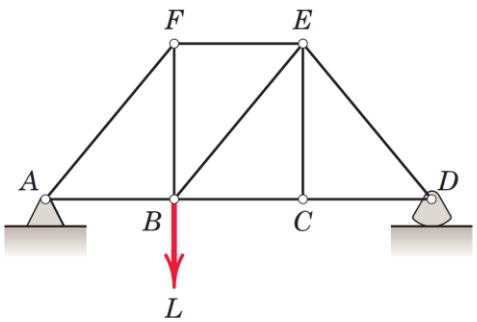


Two-Force Members

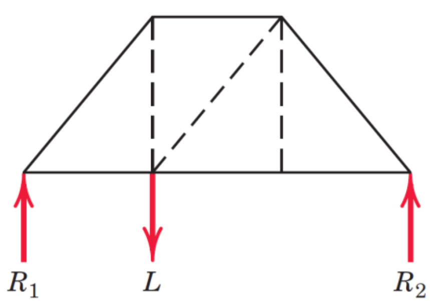


(*c*)

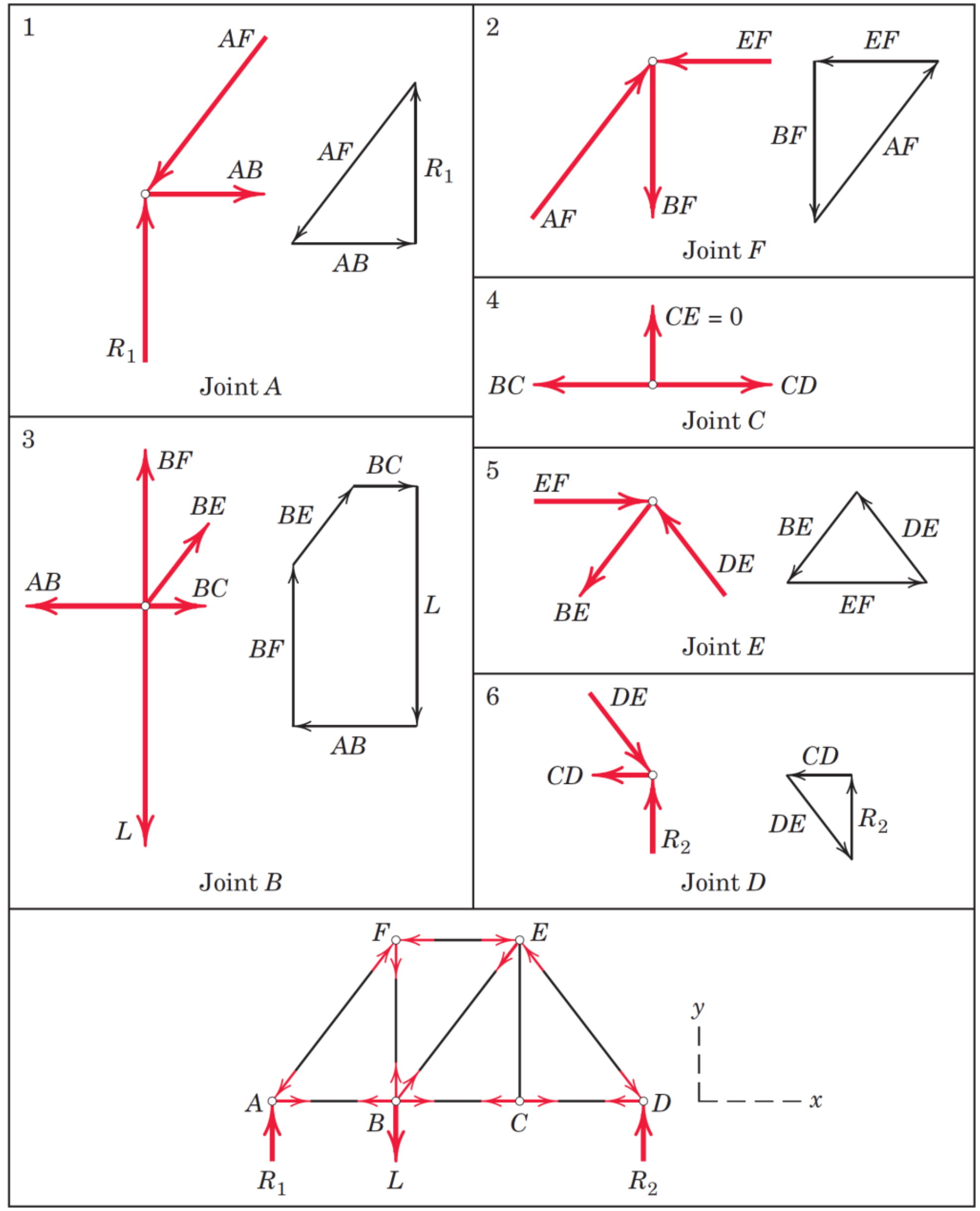
Method of Joints



(a)

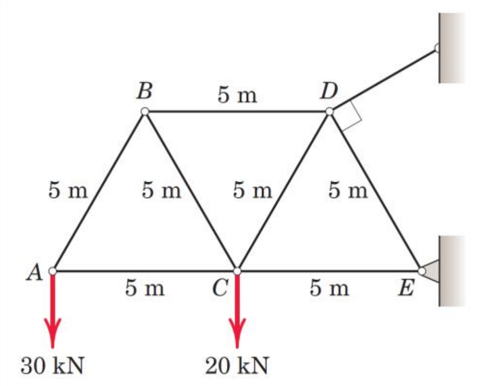


(b)



SAMPLE PROBLEM

Compute the force in each member of the loaded cantilever truss by the method of joints.

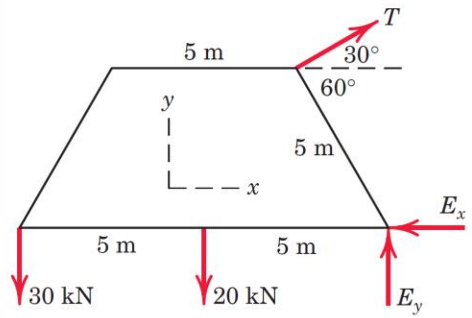


Solution. If it were not desired to calculate the external reactions at D and E, the analysis for a cantilever truss could begin with the joint at the loaded end. However, this truss will be analyzed completely, so the first step will be to compute the external forces at D and E from the free-body diagram of the truss as a whole. The equations of equilibrium give

$[\Sigma M_E = 0]$ $\qquad 5T - 20(5) - 30(10) = 0 \qquad T = 80\text{ kN}$

$[\Sigma F_x = 0]$ $\qquad 80 \cos 30° - E_x = 0 \qquad E_x = 69.3\text{ kN}$

$[\Sigma F_y = 0]$ $\qquad 80 \sin 30° + E_y - 20 - 30 = 0 \qquad E_y = 10\text{ kN}$



Next we draw free-body diagrams showing the forces acting on each of the connecting pins. The correctness of the assigned directions of the forces is verified when each joint is considered in sequence. There should be no question about the correct direction of the forces on joint A. Equilibrium requires

$[\Sigma F_y = 0]$ $\qquad 0.866AB - 30 = 0 \qquad AB = 34.6\text{ kN } T$ *Ans.*

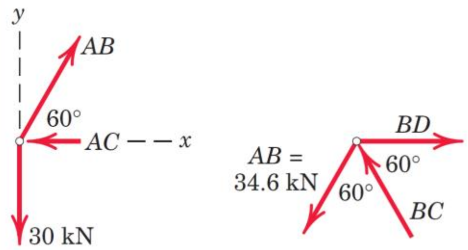
$[\Sigma F_x = 0]$ $\qquad AC - 0.5(34.6) = 0 \qquad AC = 17.32\text{ kN } C$ *Ans.*

1 where T stands for tension and C stands for compression.

Joint B must be analyzed next, since there are more than two unknown forces on joint C. The force BC must provide an upward component, in which case BD must balance the force to the left. Again the forces are obtained from

$[\Sigma F_y = 0]$ $\qquad 0.866BC - 0.866(34.6) = 0 \qquad BC = 34.6\text{ kN } C$ *Ans.*

$[\Sigma F_x = 0]$ $\qquad BD - 2(0.5)(34.6) = 0 \qquad BD = 34.6\text{ kN } T$ *Ans.*



Joint A Joint B

Joint C now contains only two unknowns, and these are found in the same way as before:

$$[\Sigma F_y = 0] \qquad 0.866CD - 0.866(34.6) - 20 = 0$$

$$CD = 57.7 \text{ kN } T \qquad \textit{Ans.}$$

$$[\Sigma F_x = 0] \qquad CE - 17.32 - 0.5(34.6) - 0.5(57.7) = 0$$

$$CE = 63.5 \text{ kN } C \qquad \textit{Ans.}$$

Finally, from joint E there results

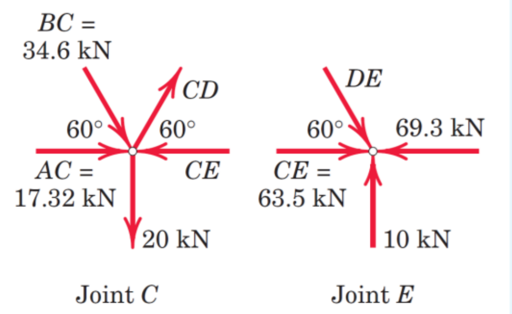
$$[\Sigma F_y = 0] \qquad 0.866DE = 10 \qquad DE = 11.55 \text{ kN } C \qquad \textit{Ans.}$$

and the equation $\Sigma F_x = 0$ checks.

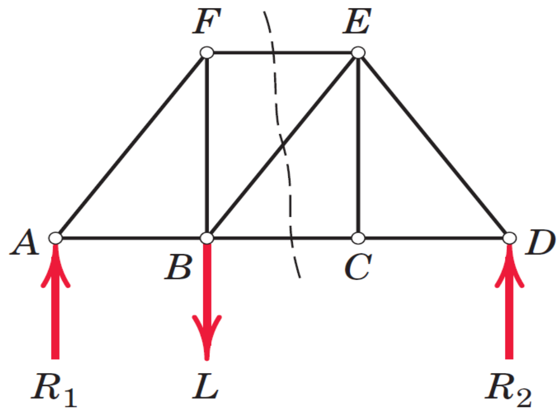
Note that the weights of the truss members have been neglected in comparison with the external loads.

Helpful Hint

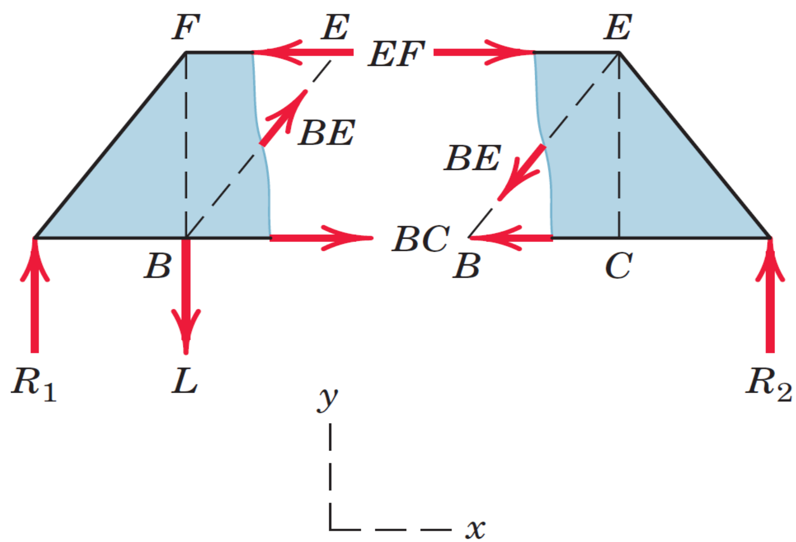
1. It should be stressed that the tension/compression designation refers to the *member*, not the joint. Note that we draw the force arrow on the same side of the joint as the member which exerts the force. In this way tension (arrow away from the joint) is distinguished from compression (arrow toward the joint).



Method of Sections



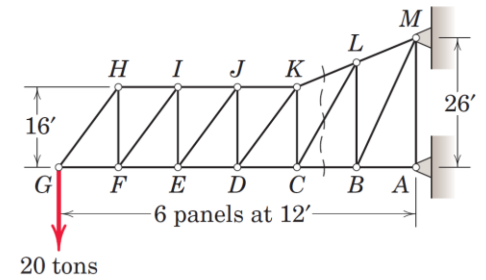
(a)

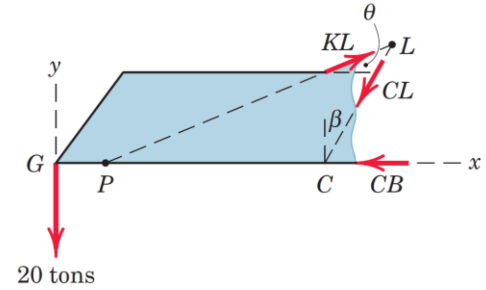


(b)

SAMPLE PROBLEM

Calculate the forces induced in members KL, CL, and CB by the 20-ton load on the cantilever truss.





Solution. Although the vertical components of the reactions at A and M are statically indeterminate with the two fixed supports, all members other than AM are statically determinate. We may pass a section directly through members KL, CL, and CB and analyze the portion of the truss to the left of this section as a
1 statically determinate rigid body.

The free-body diagram of the portion of the truss to the left of the section is shown. A moment sum about L quickly verifies the assignment of CB as compression, and a moment sum about C quickly discloses that KL is in tension. The direction of CL is not quite so obvious until we observe that KL and CB intersect at a point P to the right of G. A moment sum about P eliminates reference to KL and CB and shows that CL must be compressive to balance the moment of the 20-ton force about P. With these considerations in mind the solution becomes straightforward, as we now see how to solve for each of the three unknowns independently of the other two.

Summing moments about L requires finding the moment arm $\overline{BL} = 16 +$
2 $(26 - 16)/2 = 21$ ft. Thus,

$$[\Sigma M_L = 0] \qquad 20(5)(12) - CB(21) = 0 \qquad CB = 57.1 \text{ tons } C \qquad \textit{Ans.}$$

Next we take moments about C, which requires a calculation of $\cos\theta$. From the given dimensions we see $\theta = \tan^{-1}(5/12)$ so that $\cos\theta = 12/13$. Therefore,

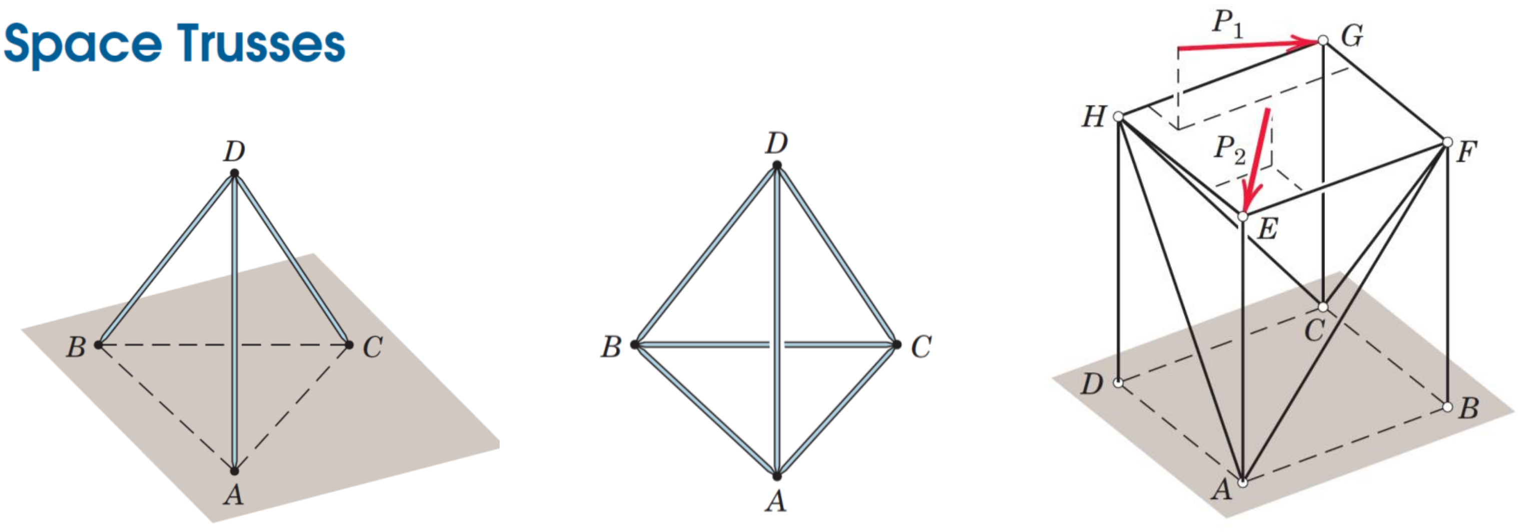
$$[\Sigma M_C = 0] \qquad 20(4)(12) - \frac{12}{13}KL(16) = 0 \qquad KL = 65 \text{ tons } T \qquad \textit{Ans.}$$

Finally, we may find CL by a moment sum about P, whose distance from C is given by $\overline{PC}/16 = 24/(26 - 16)$ or $\overline{PC} = 38.4$ ft. We also need β, which is given by $\beta = \tan^{-1}(\overline{CB}/\overline{BL}) = \tan^{-1}(12/21) = 29.7°$ and $\cos\beta = 0.868$. We now have

3 $$[\Sigma M_P = 0] \qquad 20(48 - 38.4) - CL(0.868)(38.4) = 0$$

$$CL = 5.76 \text{ tons } C \qquad \textit{Ans.}$$

Space Trusses



(a) (b) (c)

Method of Joints for Space Trusses

The method of joints developed in Art. 4/3 for plane trusses may be extended directly to space trusses by satisfying the complete vector equation

$$\Sigma \mathbf{F} = \mathbf{0}$$

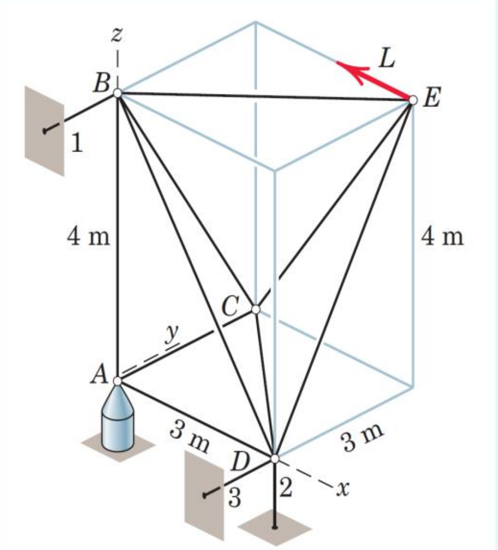
Method of Sections for Space Trusses

The method of sections developed in the previous article may also be applied to space trusses. The two vector equations

$$\Sigma \mathbf{F} = \mathbf{0} \quad \text{and} \quad \Sigma \mathbf{M} = \mathbf{0}$$

SAMPLE PROBLEM

The space truss consists of the rigid tetrahedron $ABCD$ anchored by a ball-and-socket connection at A and prevented from any rotation about the x-, y-, or z-axes by the respective links 1, 2, and 3. The load L is applied to joint E, which is rigidly fixed to the tetrahedron by the three additional links. Solve for the forces in the members at joint E and indicate the procedure for the determination of the forces in the remaining members of the truss.



Solution. We note first that the truss is supported with six properly placed constraints, which are the three at A and the links 1, 2, and 3. Also, with $m = 9$ members and $j = 5$ joints, the condition $m + 6 = 3j$ for a sufficiency of members to provide a noncollapsible structure is satisfied.

The external reactions at A, B, and D can be calculated easily as a first step, although their values will be determined from the solution of all forces on each
1 of the joints in succession.

We start with a joint on which at least one known force and not more than three unknown forces act, which in this case is joint E. The free-body diagram of joint E is shown with all force vectors arbitrarily assumed in their positive tension directions (away from the joint). The vector expressions for the three unknown forces are

2

$$\mathbf{F}_{EB} = \frac{F_{EB}}{\sqrt{2}}(-\mathbf{i} - \mathbf{j}), \qquad \mathbf{F}_{EC} = \frac{F_{EC}}{5}(-3\mathbf{i} - 4\mathbf{k}), \qquad \mathbf{F}_{ED} = \frac{F_{ED}}{5}(-3\mathbf{j} - 4\mathbf{k})$$

Equilibrium of joint E requires

$$[\Sigma\mathbf{F} = \mathbf{0}] \qquad \mathbf{L} + \mathbf{F}_{EB} + \mathbf{F}_{EC} + \mathbf{F}_{ED} = \mathbf{0} \qquad \text{or}$$

$$-L\mathbf{i} + \frac{F_{EB}}{\sqrt{2}}(-\mathbf{i} - \mathbf{j}) + \frac{F_{EC}}{5}(-3\mathbf{i} + 4\mathbf{k}) + \frac{F_{ED}}{5}(-3\mathbf{j} - 4\mathbf{k}) = \mathbf{0}$$

Rearranging terms gives

$$\left(-L - \frac{F_{EB}}{\sqrt{2}} - \frac{3F_{EC}}{5}\right)\mathbf{i} + \left(-\frac{F_{EB}}{\sqrt{2}} - \frac{3F_{ED}}{5}\right)\mathbf{j} + \left(-\frac{4F_{EC}}{5} - \frac{4F_{ED}}{5}\right)\mathbf{k} = \mathbf{0}$$

Equating the coefficients of the **i**-, **j**-, and **k**-unit vectors to zero gives the three equations

$$\frac{F_{EB}}{\sqrt{2}} + \frac{3F_{EC}}{5} = -L \qquad \frac{F_{EB}}{\sqrt{2}} + \frac{3F_{ED}}{5} = 0 \qquad F_{EC} + F_{ED} = 0$$

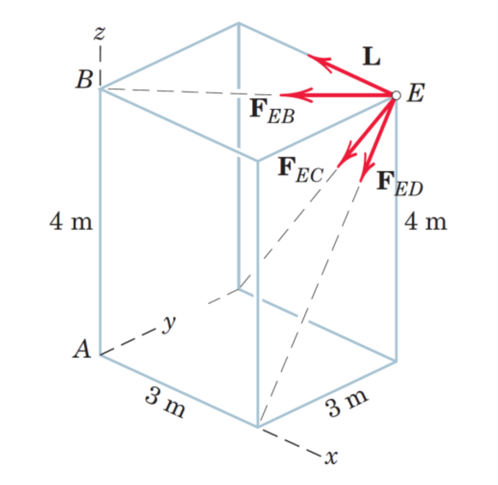
Solving the equations gives us

$$F_{EB} = -L/\sqrt{2} \qquad F_{EC} = -5L/6 \qquad F_{ED} = 5L/6 \qquad \textit{Ans.}$$

Thus, we conclude that F_{EB} and F_{EC} are compressive forces and F_{ED} is tension.

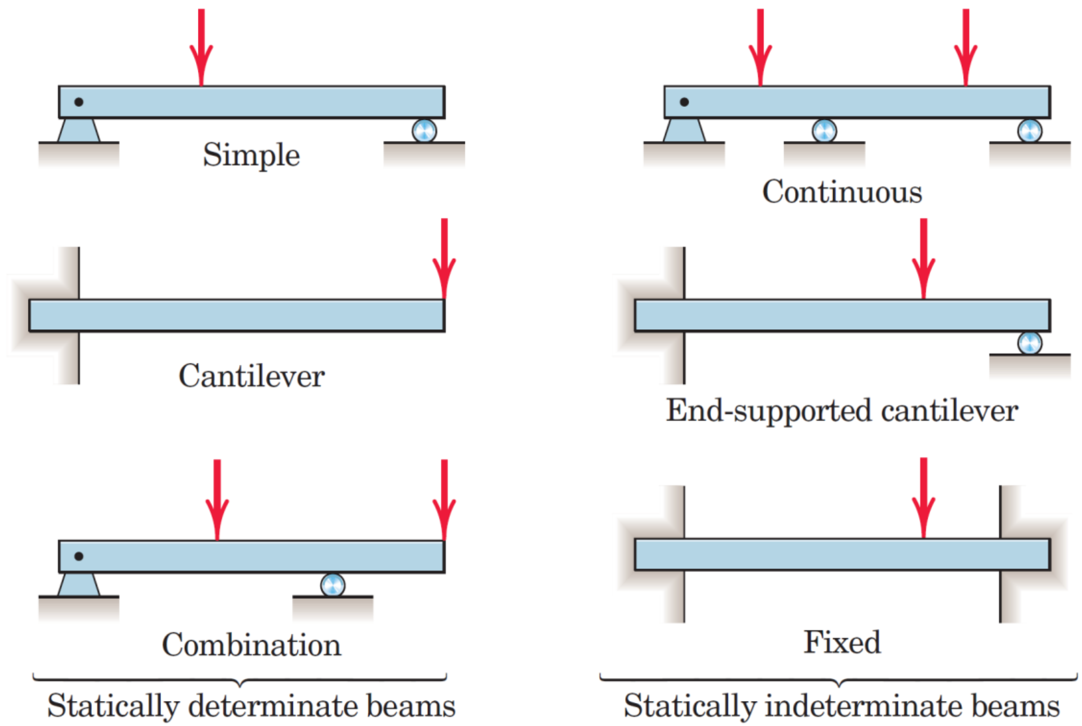
1 *Suggestion:* Draw a free-body diagram of the truss as a whole and verify that the external forces acting on the truss are $\mathbf{A}_x = L\mathbf{i}$, $\mathbf{A}_y = L\mathbf{j}$, $\mathbf{A}_z = (4L/3)\mathbf{k}$, $\mathbf{B}_y = \mathbf{0}$, $\mathbf{D}_y = -L\mathbf{j}$, $\mathbf{D}_z = -(4L/3)\mathbf{k}$.

2 With this assumption, a negative numerical value for a force indicates compression.

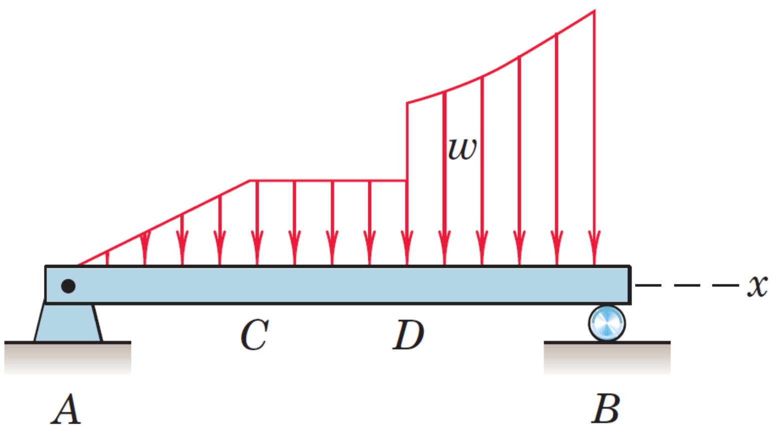


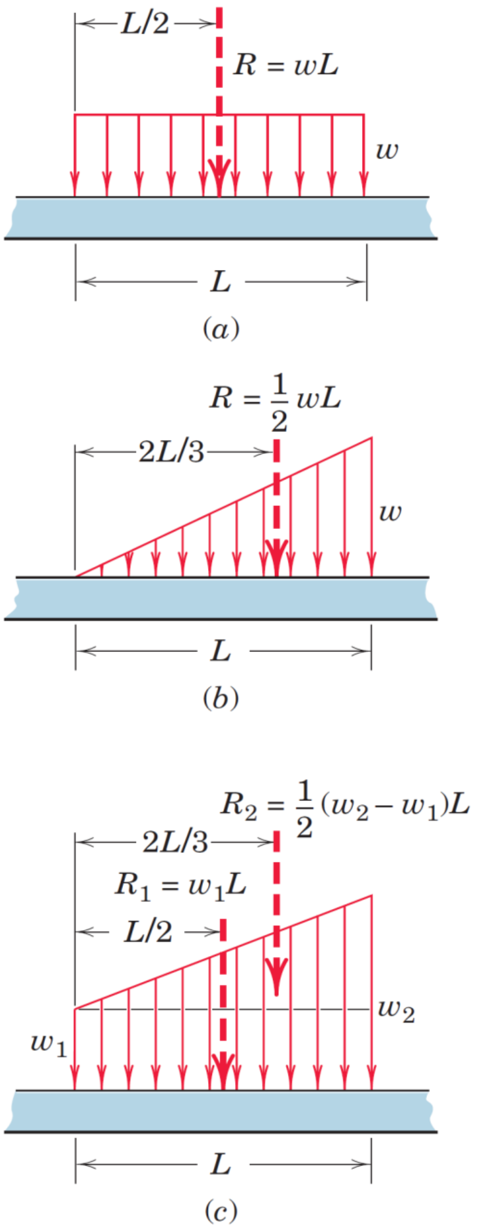
Beams

Introduction



Beams—External Effects





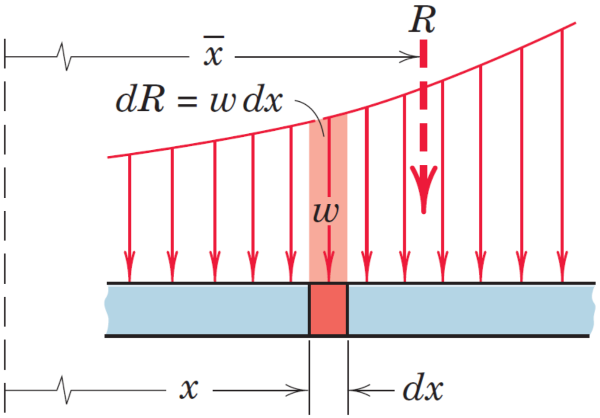
For a more general load distribution, Fig. 5/21, we must start with a differential increment of force $dR = w\ dx$. The total load R is then the sum of the differential forces, or

$$R = \int w\ dx$$

As before, the resultant R is located at the centroid of the area under consideration. The x-coordinate of this centroid is found by the principle of moments $R\bar{x} = \int xw\ dx$, or

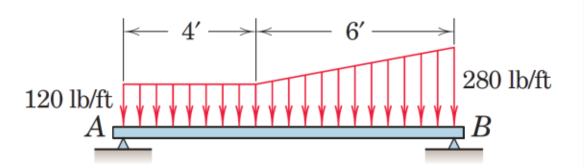
$$\bar{x} = \frac{\int xw\ dx}{R}$$

For the distribution of Fig. 5/21, the vertical coordinate of the centroid need not be found.



SAMPLE PROBLEM 5/11

Determine the equivalent concentrated load(s) and external reactions for the simply supported beam which is subjected to the distributed load shown.



Solution. The area associated with the load distribution is divided into the rectangular and triangular areas shown. The concentrated-load values are determined by computing the areas, and these loads are located at the centroids of the respective areas. (1)

Once the concentrated loads are determined, they are placed on the free-body diagram of the beam along with the external reactions at *A* and *B*. Using principles of equilibrium, we have

$[\Sigma M_A = 0]$ $\quad 1200(5) + 480(8) - R_B(10) = 0$

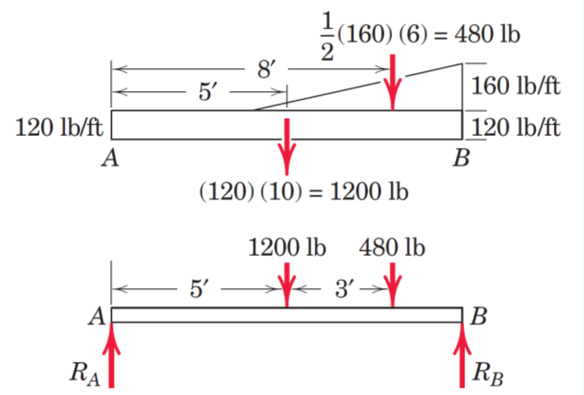
$R_B = 984 \text{ lb}$ *Ans.*

$[\Sigma M_B = 0]$ $\quad R_A(10) - 1200(5) - 480(2) = 0$

$R_A = 696 \text{ lb}$ *Ans.*

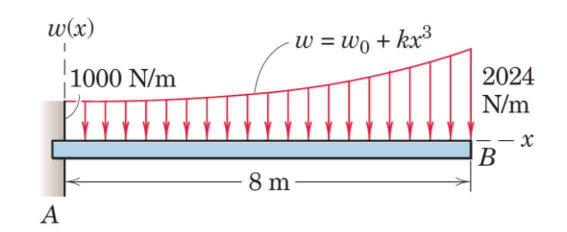
Helpful Hint

(1) Note that it is usually unnecessary to reduce a given distributed load to a *single* concentrated load.



SAMPLE PROBLEM 5/12

Determine the reaction at the support A of the loaded cantilever beam.



Solution. The constants in the load distribution are found to be $w_0 = 1000$ N/m and $k = 2$ N/m^4. The load R is then

$$R = \int w\,dx = \int_0^8 (1000 + 2x^3)\,dx = \left(1000x + \frac{x^4}{2}\right)\Bigg|_0^8 = 10\,050 \text{ N}$$

The x-coordinate of the centroid of the area is found by

$$\bar{x} = \frac{\int xw\,dx}{R} = \frac{1}{10\,050}\int_0^8 x(1000 + 2x^3)\,dx$$

$$= \frac{1}{10\,050}\left(500x^2 + \tfrac{2}{5}x^5\right)\Big|_0^8 = 4.49 \text{ m}$$

From the free-body diagram of the beam, we have

$[\Sigma M_A = 0]$ $\quad M_A - (10\,050)(4.49) = 0$

$M_A = 45\,100$ N·m *Ans.*

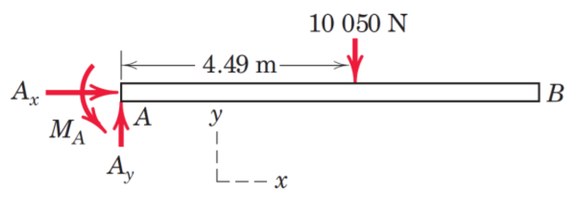
$[\Sigma F_y = 0]$ $\quad A_y = 10\,050$ N *Ans.*

Note that $A_x = 0$ by inspection.

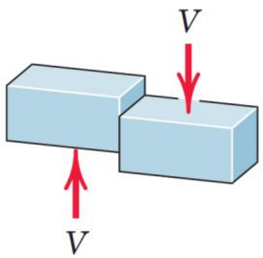
Helpful Hints

1. Use caution with the units of the constants w_0 and k.

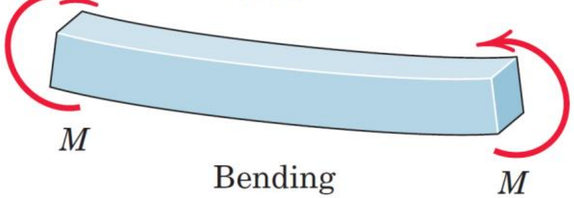
2. The student should recognize that the calculation of R and its location $\bar{x}$ is simply an application of centroids as treated in Art. 5/3.



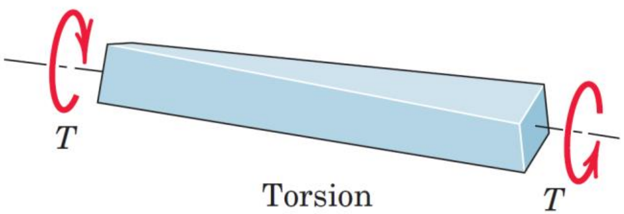
Beams—Internal Effects



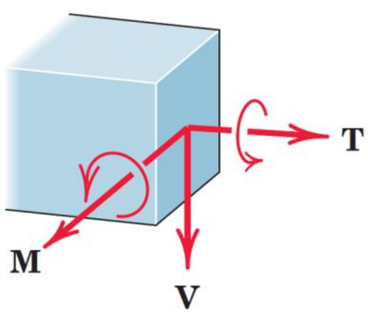
Shear



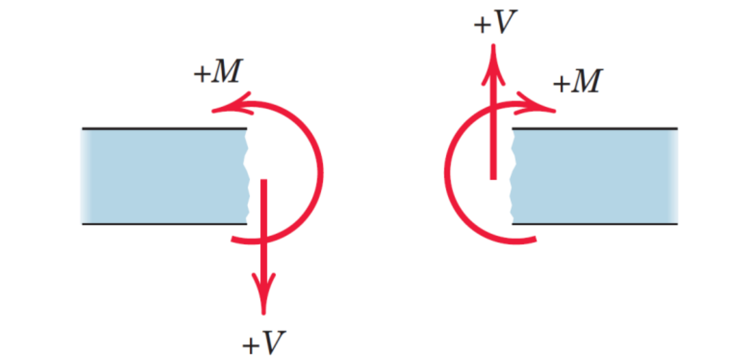
Bending

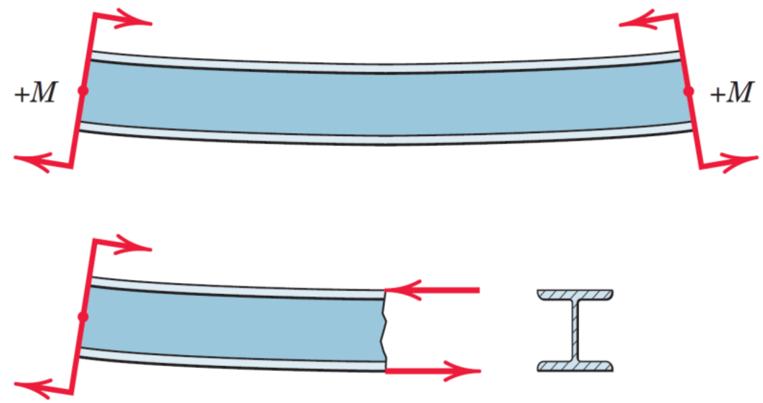


Torsion



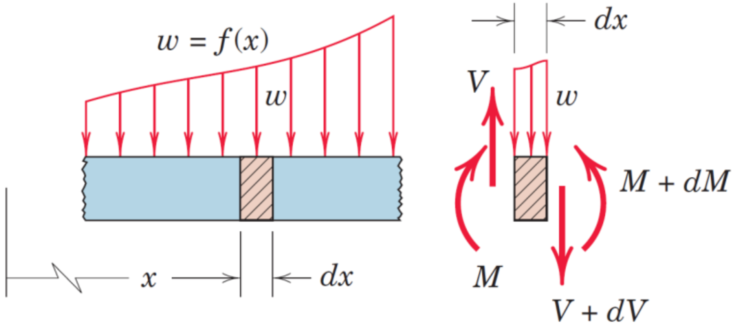
Combined loading





Equilibrium of the element requires that the sum of the vertical forces be zero. Thus, we have

$$V - w\,dx - (V + dV) = 0$$



or

$$w = -\frac{dV}{dx}$$

We see from Eq. 5/10 that the slope of the shear diagram must everywhere be equal to the negative of the value of the applied loading. Equation 5/10 holds on either side of a concentrated load but not at the concentrated load because of the discontinuity produced by the abrupt change in shear.

We may now express the shear force V in terms of the loading w by integrating Eq. 5/10. Thus,

$$\int_{V_0}^{V} dV = -\int_{x_0}^{x} w\,dx$$

or

$$V = V_0 + \text{(the negative of the area under the loading curve from } x_0 \text{ to } x)$$

In this expression V_0 is the shear force at x_0 and V is the shear force at x. Summing the area under the loading curve is usually a simple way to construct the shear-force diagram.

Equilibrium of the element in Fig. 5/25 also requires that the moment sum be zero. Summing moments about the left side of the element gives

$$M + w\,dx\,\frac{dx}{2} + (V + dV)\,dx - (M + dM) = 0$$

The two M's cancel, and the terms $w(dx)^2/2$ and $dV\,dx$ may be dropped, since they are differentials of higher order than those which remain. This leaves

$$\boxed{V = \frac{dM}{dx}} \qquad \textbf{(5/11)}$$

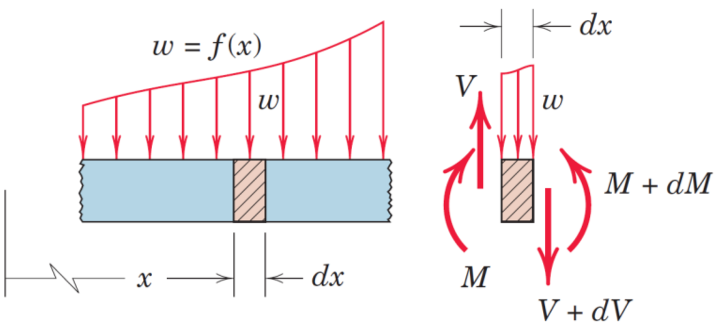
which expresses the fact that the shear everywhere is equal to the slope of the moment curve. Equation 5/11 holds on either side of a concentrated couple but not at the concentrated couple because of the discontinuity caused by the abrupt change in moment.

We may now express the moment M in terms of the shear V by integrating Eq. 5/11. Thus,

$$\int_{M_0}^{M} dM = \int_{x_0}^{x} V\,dx$$

or

$$M = M_0 + (\text{area under the shear diagram from } x_0 \text{ to } x)$$



In this expression M_0 is the bending moment at x_0 and M is the bending moment at x. For beams where there is no externally applied mo-

We observe from Eqs. 5/10 and 5/11 that the degree of V in x is one higher than that of w. Also M is of one higher degree in x than is V. Consequently, M is two degrees higher in x than w. Thus for a beam loaded by $w = kx$, which is of the first degree in x, the shear V is of the second degree in x and the bending moment M is of the third degree in x.

Equations 5/10 and 5/11 may be combined to yield

$$\boxed{\frac{d^2M}{dx^2} = -w} \qquad \textbf{(5/12)}$$

SAMPLE PROBLEM 5/13

Determine the shear and moment distributions produced in the simple beam by the 4-kN concentrated load.

Solution. From the free-body diagram of the entire beam we find the support reactions, which are

$$R_1 = 1.6 \text{ kN} \qquad R_2 = 2.4 \text{ kN}$$

A section of the beam of length x is next isolated with its free-body diagram on which we show the shear V and the bending moment M in their positive directions. Equilibrium gives

$$[\Sigma F_y = 0] \qquad 1.6 - V = 0 \qquad V = 1.6 \text{ kN}$$

$$[\Sigma M_{R_1} = 0] \qquad M - 1.6x = 0 \qquad M = 1.6x$$

1 These values of V and M apply to all sections of the beam to the left of the 4-kN load.

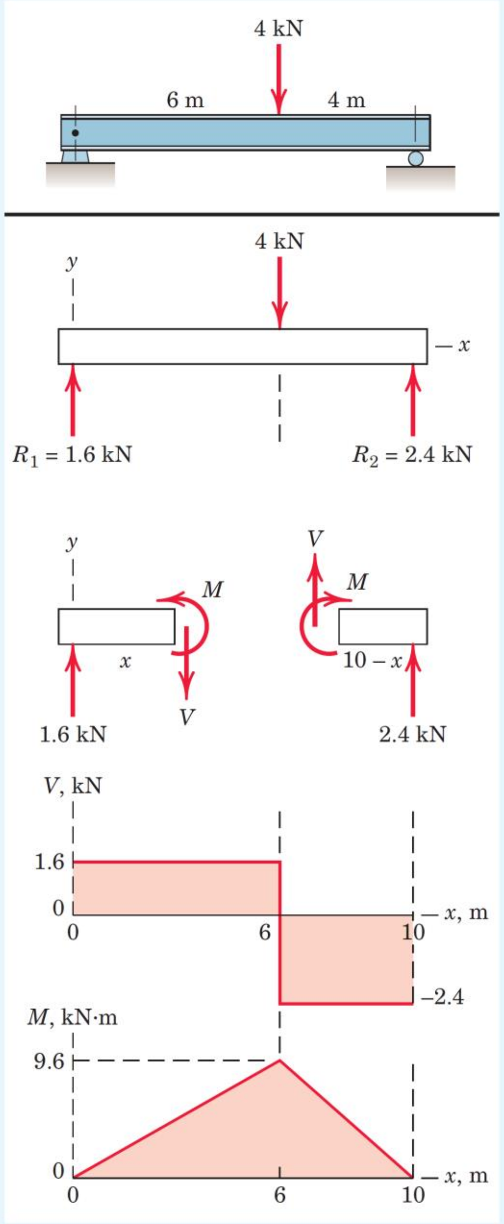
A section of the beam to the right of the 4-kN load is next isolated with its free-body diagram on which V and M are shown in their positive directions. Equilibrium requires

$$[\Sigma F_y = 0] \qquad V + 2.4 = 0 \qquad V = -2.4 \text{ kN}$$

$$[\Sigma M_{R_2} = 0] \qquad -(2.4)(10 - x) + M = 0 \qquad M = 2.4(10 - x)$$

These results apply only to sections of the beam to the right of the 4-kN load.

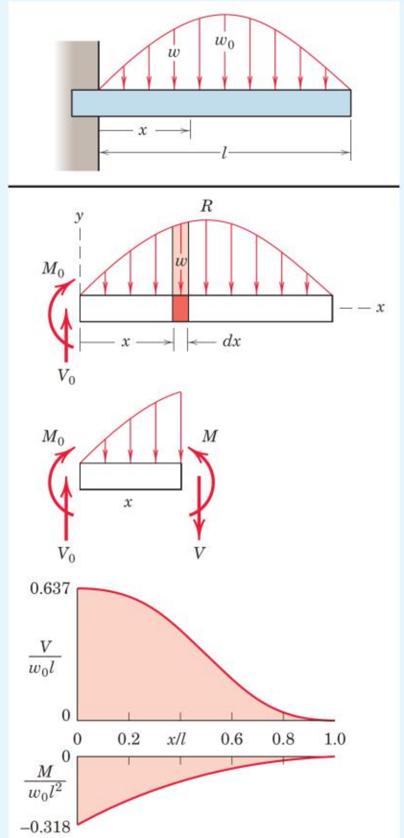
The values of V and M are plotted as shown. The maximum bending moment occurs where the shear changes direction. As we move in the positive x-direction starting with $x = 0$, we see that the moment M is merely the accumulated area under the shear diagram.



Beams

SAMPLE PROBLEM 5/14

The cantilever beam is subjected to the load intensity (force per unit length) which varies as $w = w_0 \sin(\pi x/l)$. Determine the shear force V and bending moment M as functions of the ratio x/l.



Solution. The free-body diagram of the entire beam is drawn first so that the shear force V_0 and bending moment M_0 which act at the supported end at $x = 0$ can be computed. By convention V_0 and M_0 are shown in their positive mathematical senses. A summation of vertical forces for equilibrium gives

$$[\Sigma F_y = 0] \qquad V_0 - \int_0^l w\,dx = 0 \qquad V_0 = \int_0^l w_0 \sin\frac{\pi x}{l}\,dx = \frac{2w_0 l}{\pi}$$

1 A summation of moments about the left end at $x = 0$ for equilibrium gives

$$[\Sigma M = 0] \qquad -M_0 - \int_0^l x(w\,dx) = 0 \qquad M_0 = -\int_0^l w_0 x \sin\frac{\pi x}{l}\,dx$$

$$M_0 = \frac{-w_0 l^2}{\pi^2}\left[\sin\frac{\pi x}{l} - \frac{\pi x}{l}\cos\frac{\pi x}{l}\right]_0^l = -\frac{w_0 l^2}{\pi}$$

From a free-body diagram of an arbitrary section of length x, integration of Eq. 5/10 permits us to find the shear force internal to the beam. Thus,

2 $$[dV = -w\,dx] \qquad \int_{V_0}^{V} dV = -\int_0^x w_0 \sin\frac{\pi x}{l}\,dx$$

$$V - V_0 = \left[\frac{w_0 l}{\pi}\cos\frac{\pi x}{l}\right]_0^x \qquad V - \frac{2w_0 l}{\pi} = \frac{w_0 l}{\pi}\left(\cos\frac{\pi x}{l} - 1\right)$$

or in dimensionless form

$$\frac{V}{w_0 l} = \frac{1}{\pi}\left(1 + \cos\frac{\pi x}{l}\right) \qquad \textit{Ans.}$$

The bending moment is obtained by integration of Eq. 5/11, which gives

$$[dM = V\,dx] \qquad \int_{M_0}^{M} dM = \int_0^x \frac{w_0 l}{\pi}\left(1 + \cos\frac{\pi x}{l}\right)dx$$

$$M - M_0 = \frac{w_0 l}{\pi}\left[x + \frac{l}{\pi}\sin\frac{\pi x}{l}\right]_0^x$$

$$M = -\frac{w_0 l^2}{\pi} + \frac{w_0 l}{\pi}\left[x + \frac{l}{\pi}\sin\frac{\pi x}{l} - 0\right]$$

or in dimensionless form

$$\frac{M}{w_0 l^2} = \frac{1}{\pi}\left(\frac{x}{l} - 1 + \frac{1}{\pi}\sin\frac{\pi x}{l}\right) \qquad \textit{Ans.}$$

The variations of $V/w_0 l$ and $M/w_0 l^2$ with x/l are shown in the bottom figures. The negative values of $M/w_0 l^2$ indicate that physically the bending moment is in the direction opposite to that shown.

Helpful Hints

1 In this case of symmetry it is clear that the resultant $R = V_0 = 2w_0 l/\pi$ of the load distribution acts at midspan, so that the moment requirement is simply $M_0 = -Rl/2 = -w_0 l^2/\pi$. The minus sign tells us that physically the bending moment at $x = 0$ is opposite to that represented on the free-body diagram.

2 The free-body diagram serves to remind us that the integration limits for V as well as for x must be accounted for. We see that the expression for V is positive, so that the shear force is as represented on the free-body diagram.

SAMPLE PROBLEM 5/15

Draw the shear-force and bending-moment diagrams for the loaded beam and determine the maximum moment M and its location x from the left end.

Solution. The support reactions are most easily obtained by considering the resultants of the distributed loads as shown on the free-body diagram of the beam as a whole. The first interval of the beam is analyzed from the free-body diagram of the section for $0 < x < 4$ ft. A summation of vertical forces and a moment summation about the cut section yield

$$[\Sigma F_y = 0] \qquad V = 247 - 12.5x^2$$

$$[\Sigma M = 0] \qquad M + (12.5x^2)\frac{x}{3} - 247x = 0 \qquad M = 247x - 4.17x^3$$

These values of V and M hold for $0 < x < 4$ ft and are plotted for that interval in the shear and moment diagrams shown.

From the free-body diagram of the section for which $4 < x < 8$ ft, equilibrium in the vertical direction and a moment sum about the cut section give

$$[\Sigma F_y = 0] \qquad V + 100(x - 4) + 200 - 247 = 0 \qquad V = 447 - 100x$$

$$[\Sigma M = 0] \qquad M + 100(x - 4)\frac{x - 4}{2} + 200[x - \tfrac{2}{3}(4)] - 247x = 0$$

$$M = -267 + 447x - 50x^2$$

These values of V and M are plotted on the shear and moment diagrams for the interval $4 < x < 8$ ft.

The analysis of the remainder of the beam is continued from the free-body diagram of the portion of the beam to the right of a section in the next interval. It should be noted that V and M are represented in their positive directions. A vertical-force summation and a moment summation about the section yield

$$V = -353 \text{ lb} \qquad \text{and} \qquad M = 2930 - 353x$$

These values of V and M are plotted on the shear and moment diagrams for the interval $8 < x < 10$ ft.

The last interval may be analyzed by inspection. The shear is constant at +300 lb, and the moment follows a straight-line relation beginning with zero at the right end of the beam.

The maximum moment occurs at $x = 4.47$ ft, where the shear curve crosses the zero axis, and the magnitude of M is obtained for this value of x by substitution into the expression for M for the second interval. The maximum moment is

$$M = 732 \text{ lb-ft} \qquad \textit{Ans.}$$

As before, note that the change in moment M up to any section equals the area under the shear diagram up to that section. For instance, for $x < 4$ ft,

$$\left[\Delta M = \int V\,dx\right] \qquad M - 0 = \int_0^x (247 - 12.5x^2)\,dx$$

and, as above, $\qquad M = 247x - 4.17x^3$