In the name of God

Mechanics of Materials

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Concept of Stress

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The force per unit area, or intensity of the forces distributed over a given section, is called the *stress* on that section and is denoted by the Greek letter σ (sigma). The stress in a member of cross-sectional area A subjected to an axial load **P** (Fig. 1.8) is therefore obtained by dividing the magnitude P of the load by the area A:

$$\sigma = \frac{P}{A} \tag{1.5}$$

A positive sign will be used to indicate a tensile stress (member in tension) and a negative sign to indicate a compressive stress (member in compression).

Since SI metric units are used in this discussion, with P expressed in newtons (N) and A in square meters (m^2), the stress σ will be expressed in N/m^2. This unit is called a *pascal* (Pa). However, one finds that the pascal is an exceedingly small quantity and that, in practice, multiples of this unit must be used, namely, the kilopascal (kPa), the megapascal (MPa), and the gigapascal (GPa). We have

$$1 \text{ kPa} = 10^3 \text{ Pa} = 10^3 \text{ N/m}^2$$

$$1 \text{ MPa} = 10^6 \text{ Pa} = 10^6 \text{ N/m}^2$$

$$1 \text{ GPa} = 10^9 \text{ Pa} = 10^9 \text{ N/m}^2$$

When U.S. customary units are used, the force P is usually expressed in pounds (lb) or kilopounds (kip), and the cross-sectional area A in square inches (in^2). The stress σ will then be expressed in pounds per square inch (psi) or kilopounds per square inch (ksi).†

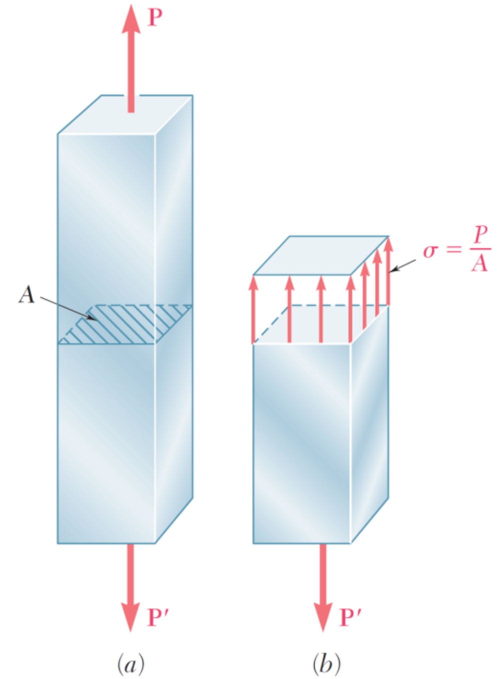


Fig. 1.8 Member with an axial load.

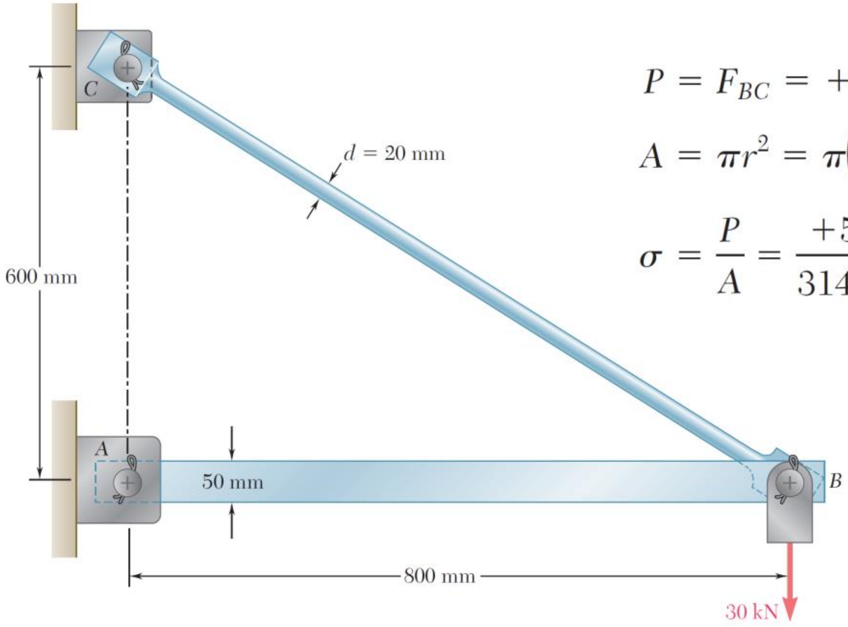


Fig. 1.1 Boom used to support a 30-kN load.

$$P = F_{BC} = +50 \text{ kN} = +50 \times 10^3 \text{ N}$$

$$A = \pi r^2 = \pi\left(\frac{20 \text{ mm}}{2}\right)^2 = \pi(10 \times 10^{-3} \text{ m})^2 = 314 \times 10^{-6} \text{ m}^2$$

$$\sigma = \frac{P}{A} = \frac{+50 \times 10^3 \text{ N}}{314 \times 10^{-6} \text{ m}^2} = +159 \times 10^6 \text{ Pa} = +159 \text{ MPa}$$

$$\sigma_{\text{all}} = \frac{P}{A} \qquad A = \frac{P}{\sigma_{\text{all}}} = \frac{50 \times 10^3 \text{ N}}{100 \times 10^6 \text{ Pa}} = 500 \times 10^{-6} \text{ m}^2$$

and, since $A = \pi r^2$,

$$r = \sqrt{\frac{A}{\pi}} = \sqrt{\frac{500 \times 10^{-6} \text{ m}^2}{\pi}} = 12.62 \times 10^{-3} \text{ m} = 12.62 \text{ mm}$$

$$d = 2r = 25.2 \text{ mm}$$

Concept of Stress

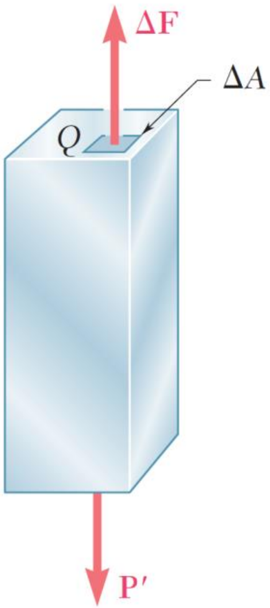


Fig. 1.9

$$\sigma = \lim_{\Delta A \to 0} \frac{\Delta F}{\Delta A}$$

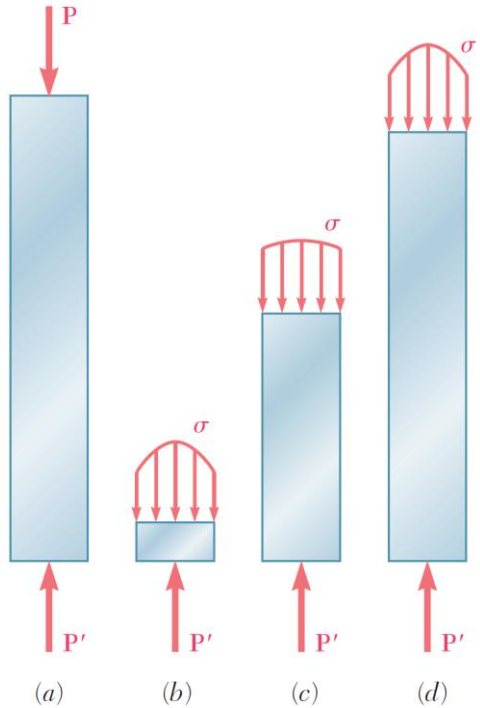


Fig. 1.10 Stress distributions at different sections along axially loaded member.

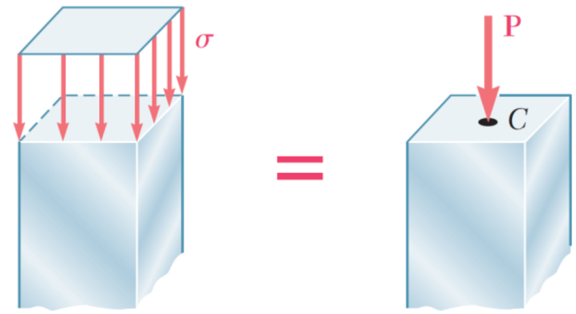


Fig. 1.11

$$\int dF = \int_A \sigma \, dA$$

But the conditions of equilibrium of each of the portions of rod shown in Fig. 1.10 require that this magnitude be equal to the magnitude P of the concentrated loads. We have, therefore,

$$P = \int dF = \int_A \sigma \, dA \tag{1.7}$$

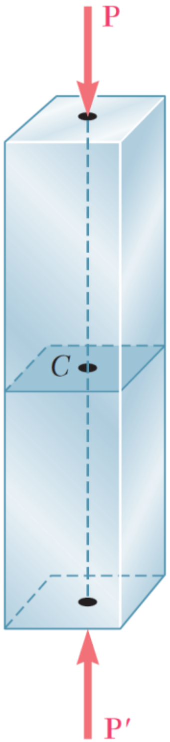
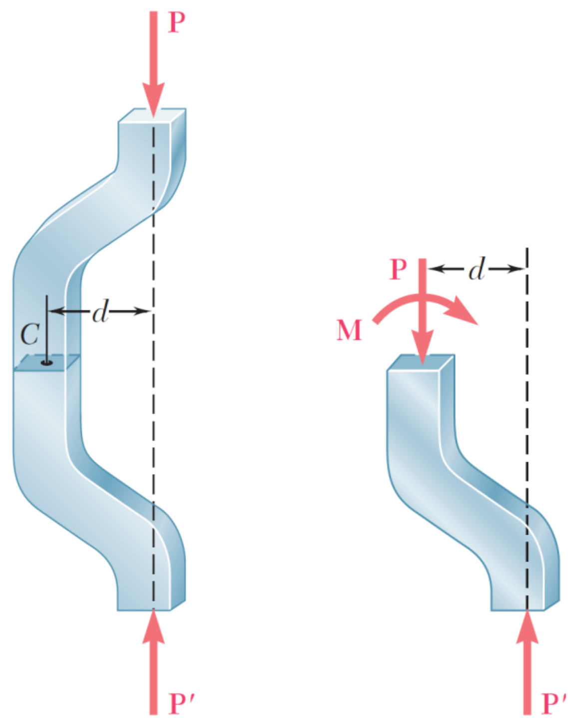


Fig. 1.12

Fig. 1.13 Eccentric axial loading.



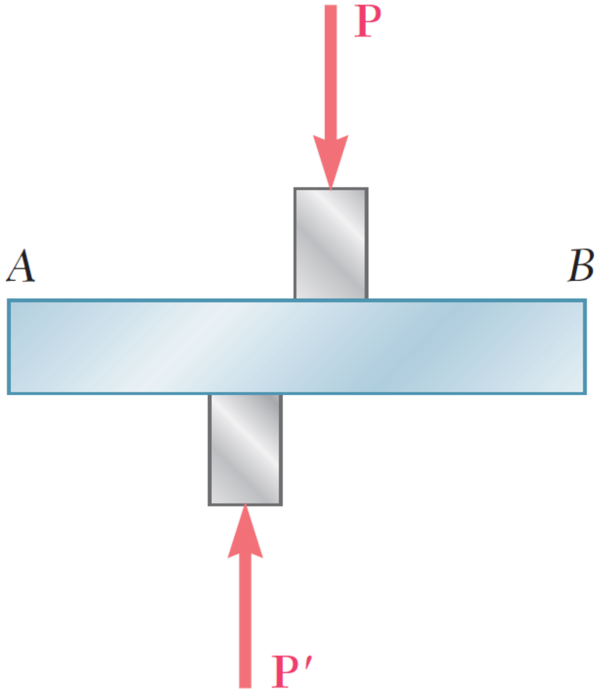


Fig. 1.14 Member with transverse loads.

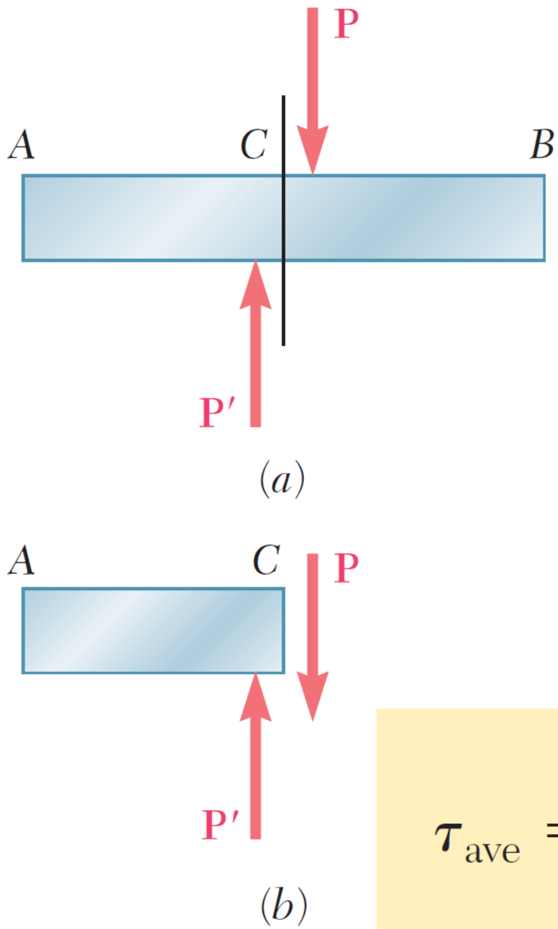


Fig. 1.15

$$\tau_{\text{ave}} = \frac{P}{A}$$

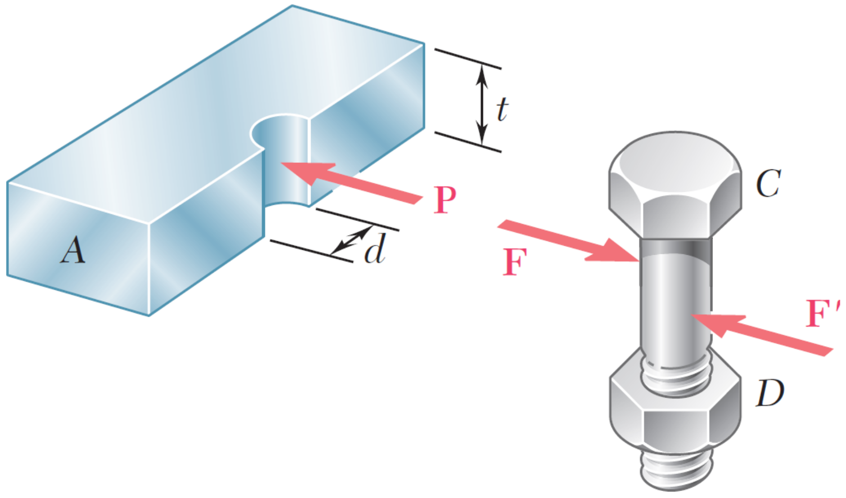


Fig. 1.20

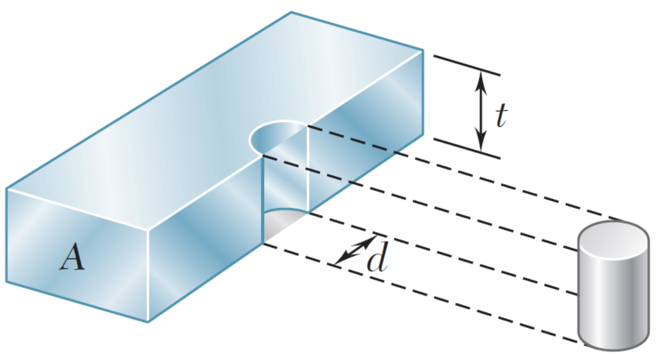
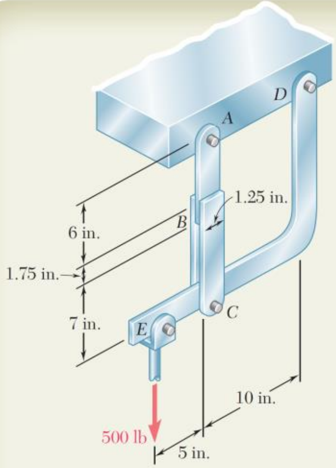


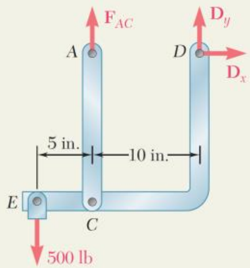
Fig. 1.21

$$\sigma_b = \frac{P}{A} = \frac{P}{td}$$



SAMPLE PROBLEM 1.1

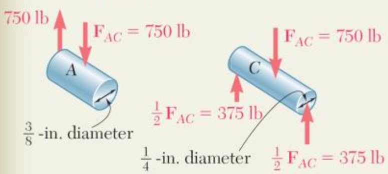
In the hanger shown, the upper portion of link ABC is $\frac{3}{8}$ in. thick and the lower portions are each $\frac{1}{4}$ in. thick. Epoxy resin is used to bond the upper and lower portions together at B. The pin at A is of $\frac{3}{8}$-in. diameter while a $\frac{1}{4}$-in.-diameter pin is used at C. Determine (*a*) the shearing stress in pin A, (*b*) the shearing stress in pin C, (*c*) the largest normal stress in link ABC, (*d*) the average shearing stress on the bonded surfaces at B, (*e*) the bearing stress in the link at C.



SOLUTION

Free Body: Entire Hanger. Since the link ABC is a two-force member, the reaction at A is vertical; the reaction at D is represented by its components $\mathbf{D}_x$ and $\mathbf{D}_y$. We write

$$+\circlearrowleft \Sigma M_D = 0: \qquad (500 \text{ lb})(15 \text{ in.}) - F_{AC}(10 \text{ in.}) = 0$$
$$F_{AC} = +750 \text{ lb} \qquad F_{AC} = 750 \text{ lb} \quad \textit{tension}$$

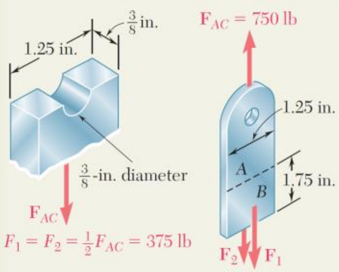


a. Shearing Stress in Pin *A*. Since this $\frac{3}{8}$-in.-diameter pin is in single shear, we write

$$\tau_A = \frac{F_{AC}}{A} = \frac{750 \text{ lb}}{\frac{1}{4}\pi(0.375 \text{ in.})^2} \qquad \tau_A = 6790 \text{ psi} \blacktriangleleft$$

b. Shearing Stress in Pin *C*. Since this $\frac{1}{4}$-in.-diameter pin is in double shear, we write

$$\tau_C = \frac{\frac{1}{2}F_{AC}}{A} = \frac{375 \text{ lb}}{\frac{1}{4}\pi(0.25 \text{ in.})^2} \qquad \tau_C = 7640 \text{ psi} \blacktriangleleft$$

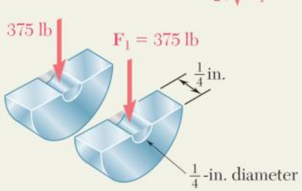


c. Largest Normal Stress in Link *ABC*. The largest stress is found where the area is smallest; this occurs at the cross section at A where the $\frac{3}{8}$-in. hole is located. We have

$$\sigma_A = \frac{F_{AC}}{A_{\text{net}}} = \frac{750 \text{ lb}}{(\frac{3}{8} \text{ in.})(1.25 \text{ in.} - 0.375 \text{ in.})} = \frac{750 \text{ lb}}{0.328 \text{ in}^2} \qquad \sigma_A = 2290 \text{ psi} \blacktriangleleft$$

d. Average Shearing Stress at *B*. We note that bonding exists on both sides of the upper portion of the link and that the shear force on each side is $F_1 = (750 \text{ lb})/2 = 375$ lb. The average shearing stress on each surface is thus

$$\tau_B = \frac{F_1}{A} = \frac{375 \text{ lb}}{(1.25 \text{ in.})(1.75 \text{ in.})} \qquad \tau_B = 171.4 \text{ psi} \blacktriangleleft$$



e. Bearing Stress in Link at *C*. For each portion of the link, $F_1 = 375$ lb and the nominal bearing area is $(0.25 \text{ in.})(0.25 \text{ in.}) = 0.0625 \text{ in}^2$.

$$\sigma_b = \frac{F_1}{A} = \frac{375 \text{ lb}}{0.0625 \text{ in}^2} \qquad \sigma_b = 6000 \text{ psi} \blacktriangleleft$$

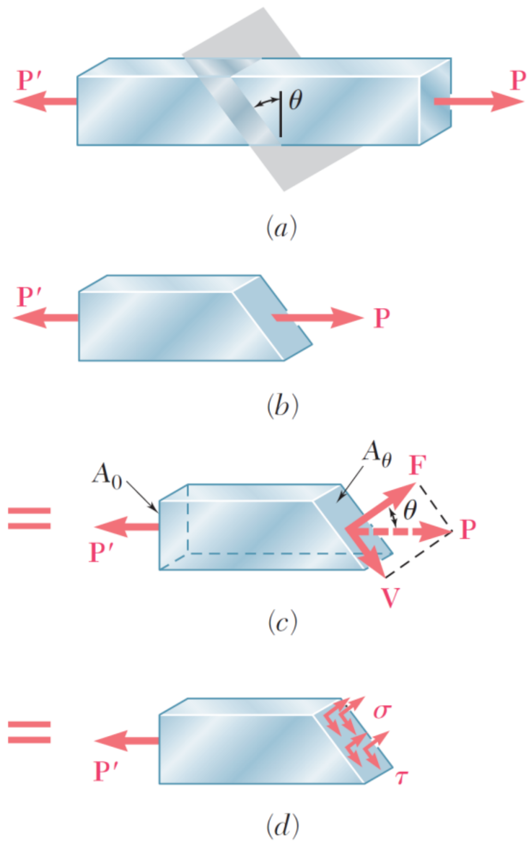


Fig. 1.28

Consider the two-force member of Fig. 1.26, which is subjected to axial forces **P** and **P**′. If we pass a section forming an angle θ with a normal plane (Fig. 1.28*a*) and draw the free-body diagram of the portion of member located to the left of that section (Fig. 1.28*b*), we find from the equilibrium conditions of the free body that the distributed forces acting on the section must be equivalent to the force **P.**

Resolving **P** into components **F** and **V,** respectively normal and tangential to the section (Fig. 1.28*c*), we have

$$F = P\cos\theta \qquad V = P\sin\theta \tag{1.12}$$

The force **F** represents the resultant of normal forces distributed over the section, and the force **V** the resultant of shearing forces (Fig. 1.28*d*). The average values of the corresponding normal and shearing stresses are obtained by dividing, respectively, F and V by the area A_θ of the section:

$$\sigma = \frac{F}{A_\theta} \qquad \tau = \frac{V}{A_\theta} \tag{1.13}$$

Substituting for F and V from (1.12) into (1.13), and observing from Fig. 1.28*c* that $A_0 = A_\theta \cos\theta$, or $A_\theta = A_0/\cos\theta$, where A_0 denotes the area of a section perpendicular to the axis of the member, we obtain

$$\sigma = \frac{P\cos\theta}{A_0/\cos\theta} \qquad \tau = \frac{P\sin\theta}{A_0/\cos\theta}$$

or

$$\sigma = \frac{P}{A_0}\cos^2\theta \qquad \tau = \frac{P}{A_0}\sin\theta\cos\theta \tag{1.14}$$

We note from the first of Eqs. (1.14) that the normal stress σ is maximum when $\theta = 0$, i.e., when the plane of the section is perpendicular to the axis of the member, and that it approaches zero as θ approaches 90°. We check that the value of σ when $\theta = 0$ is

$$\sigma_m = \frac{P}{A_0} \tag{1.15}$$

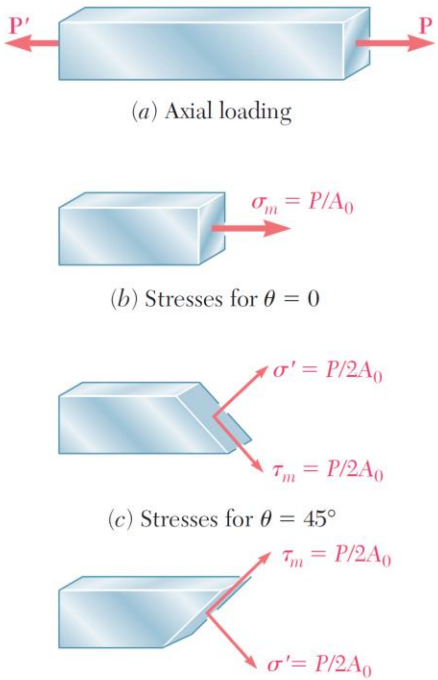
as we found earlier in Sec. 1.3. The second of Eqs. (1.14) shows that the shearing stress τ is zero for $\theta = 0$ and $\theta = 90°$, and that for $\theta = 45°$ it reaches its maximum value

$$\tau_m = \frac{P}{A_0}\sin 45°\cos 45° = \frac{P}{2A_0} \tag{1.16}$$

The first of Eqs. (1.14) indicates that, when $\theta = 45°$, the normal stress σ' is also equal to $P/2A_0$:

$$\sigma' = \frac{P}{A_0}\cos^2 45° = \frac{P}{2A_0} \tag{1.17}$$

The results obtained in Eqs. (1.15), (1.16), and (1.17) are shown graphically in Fig. 1.29. We note that the same loading may produce either a normal stress $\sigma_m = P/A_0$ and no shearing stress (Fig. 1.29*b*), or a normal and a shearing stress of the same magnitude $\sigma' = \tau_m = P/2A_0$ (Fig. 1.29 *c* and *d*), depending upon the orientation of the section.



(*d*) Stresses for $\theta = -45°$

Fig. 1.29

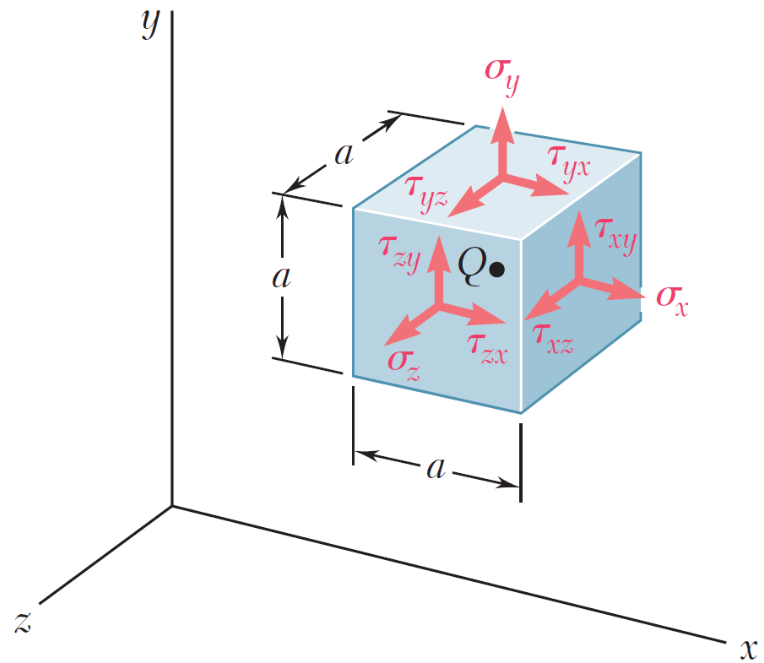


Fig. 1.34

$$\tau_{xy} = \tau_{yx}$$

$$\tau_{yz} = \tau_{zy}$$

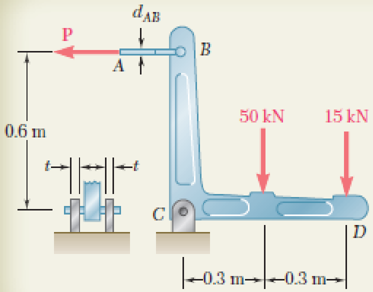
$$\tau_{zx} = \tau_{xz}$$

the *ultimate strength in tension* of the material, is

$$\sigma_U = \frac{P_U}{A}$$

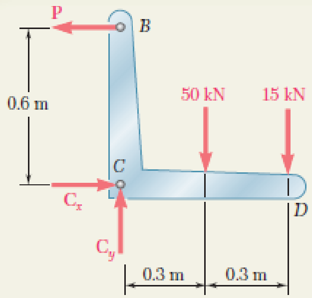
$$\text{Factor of safety} = F.S. = \frac{\text{ultimate load}}{\text{allowable load}}$$

$$\text{Factor of safety} = F.S. = \frac{\text{ultimate stress}}{\text{allowable stress}}$$



SAMPLE PROBLEM 1.3

Two forces are applied to the bracket *BCD* as shown. (*a*) Knowing that the control rod *AB* is to be made of a steel having an ultimate normal stress of 600 MPa, determine the diameter of the rod for which the factor of safety with respect to failure will be 3.3. (*b*) The pin at *C* is to be made of a steel having an ultimate shearing stress of 350 MPa. Determine the diameter of the pin *C* for which the factor of safety with respect to shear will also be 3.3. (*c*) Determine the required thickness of the bracket supports at *C* knowing that the allowable bearing stress of the steel used is 300 MPa.



SOLUTION

Free Body: Entire Bracket. The reaction at *C* is represented by its components $\mathbf{C}_x$ and $\mathbf{C}_y$.

$+\circlearrowleft \Sigma M_C = 0$: $\quad P(0.6 \text{ m}) - (50 \text{ kN})(0.3 \text{ m}) - (15 \text{ kN})(0.6 \text{ m}) = 0 \quad P = 40 \text{ kN}$

$\Sigma F_x = 0$: $\quad C_x = 40 \text{ k}$

$\Sigma F_y = 0$: $\quad C_y = 65 \text{ kN} \qquad C = \sqrt{C_x^2 + C_y^2} = 76.3 \text{ kN}$

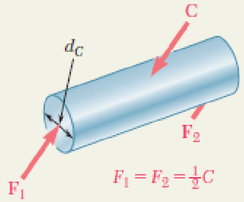
a. Control Rod *AB*. Since the factor of safety is to be 3.3, the allowable stress is

$$\sigma_{\text{all}} = \frac{\sigma_U}{F.S.} = \frac{600 \text{ MPa}}{3.3} = 181.8 \text{ MPa}$$

For $P = 40$ kN the cross-sectional area required is

$$A_{\text{req}} = \frac{P}{\sigma_{\text{all}}} = \frac{40 \text{ kN}}{181.8 \text{ MPa}} = 220 \times 10^{-6} \text{ m}^2$$

$$A_{\text{req}} = \frac{\pi}{4} d_{AB}^2 = 220 \times 10^{-6} \text{ m}^2 \qquad d_{AB} = 16.74 \text{ mm} \blacktriangleleft$$



b. Shear in Pin *C*. For a factor of safety of 3.3, we have

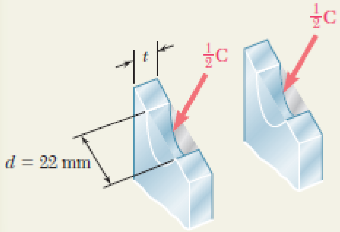
$$\tau_{\text{all}} = \frac{\tau_U}{F.S.} = \frac{350 \text{ MPa}}{3.3} = 106.1 \text{ MPa}$$

Since the pin is in double shear, we write

$$A_{\text{req}} = \frac{C/2}{\tau_{\text{all}}} = \frac{(76.3 \text{ kN})/2}{106.1 \text{ MPa}} = 360 \text{ mm}^2$$

$$A_{\text{req}} = \frac{\pi}{4} d_C^2 = 360 \text{ mm}^2 \qquad d_C = 21.4 \text{ mm} \qquad \text{Use: } d_C = 22 \text{ mm} \blacktriangleleft$$

The next larger size pin available is of 22-mm diameter and should be used.



c. Bearing at *C*. Using $d = 22$ mm, the nominal bearing area of each bracket is $22t$. Since the force carried by each bracket is $C/2$ and the allowable bearing stress is 300 MPa, we write

$$A_{\text{req}} = \frac{C/2}{\sigma_{\text{all}}} = \frac{(76.3 \text{ kN})/2}{300 \text{ MPa}} = 127.2 \text{ mm}^2$$

Thus $22t = 127.2 \qquad t = 5.78 \text{ mm} \qquad \text{Use: } t = 6 \text{ mm} \blacktriangleleft$

Stress and Strain-
Axial Loading

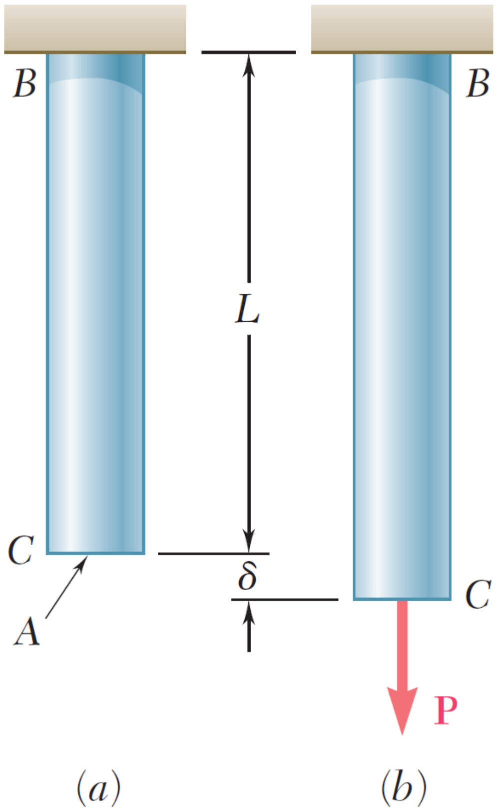


Fig. 2.1 Deformation of axially-loaded rod.

$$\epsilon = \frac{\delta}{L}$$

Stress and Strain—Axial Loading

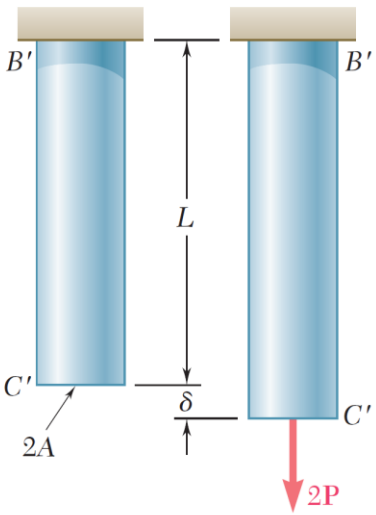


Fig. 2.3

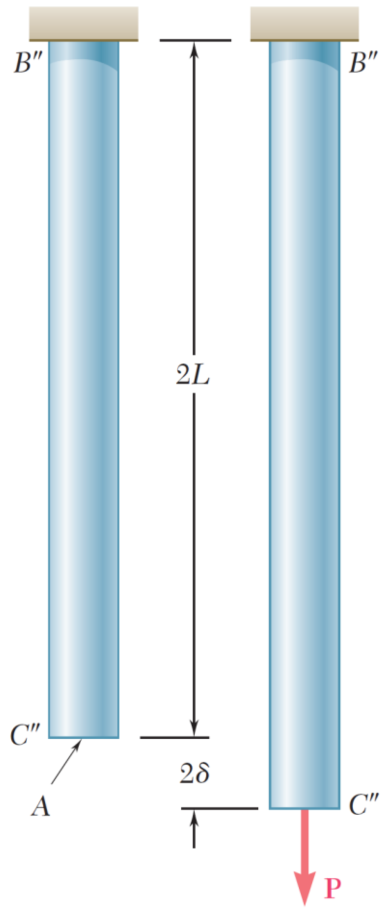


Fig. 2.4

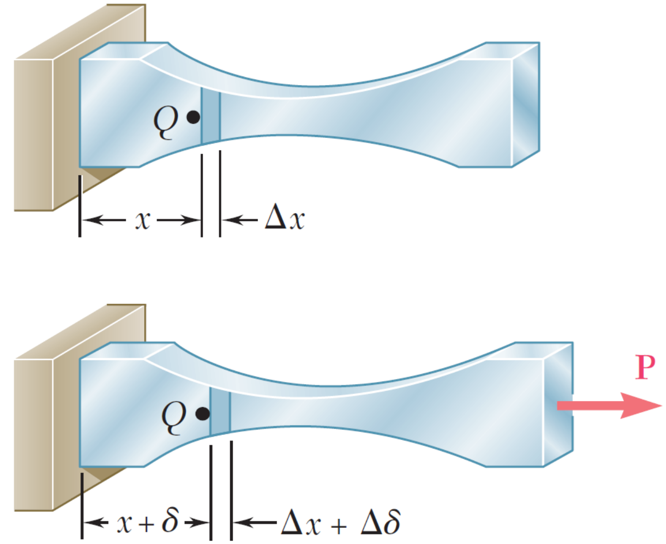


Fig. 2.5 Deformation of axially-loaded member of variable cross-sectional area.

$$\epsilon = \lim_{\Delta x \to 0} \frac{\Delta \delta}{\Delta x} = \frac{d\delta}{dx}$$

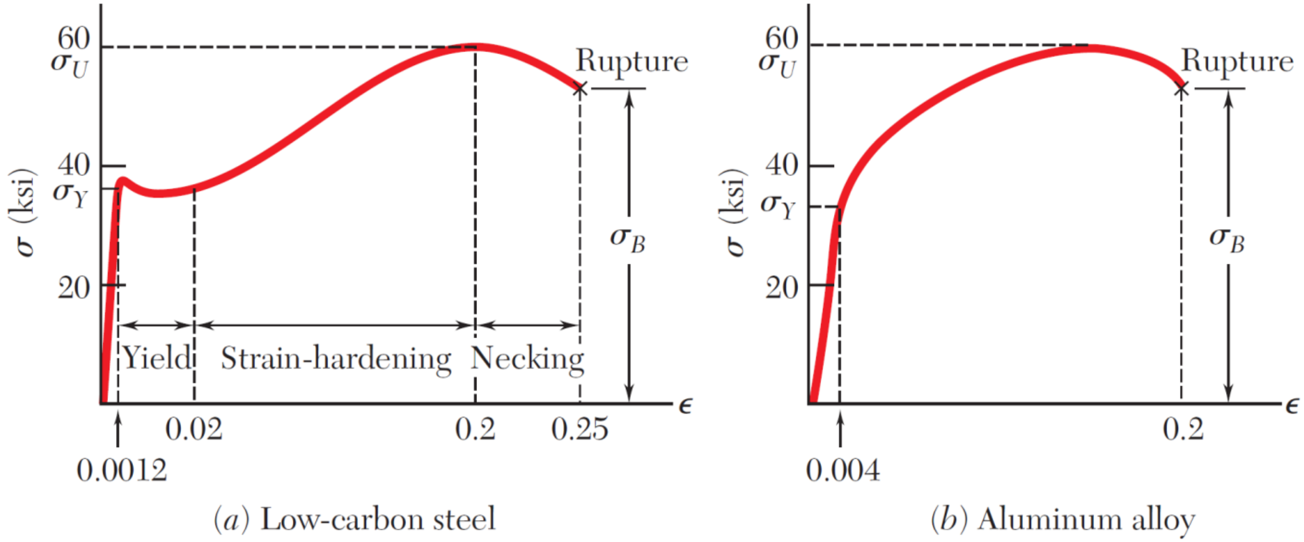


Fig. 2.6 Stress-strain diagrams of two typical ductile materials.

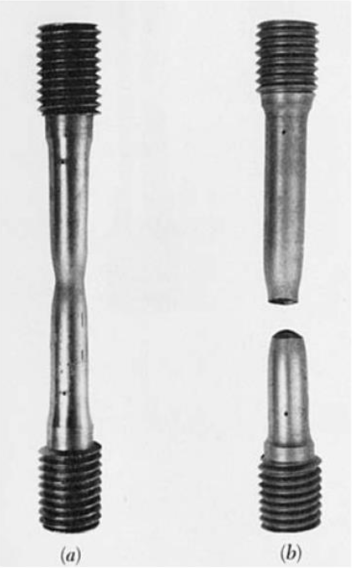


Photo 2.4 Tested specimen of a ductile material.

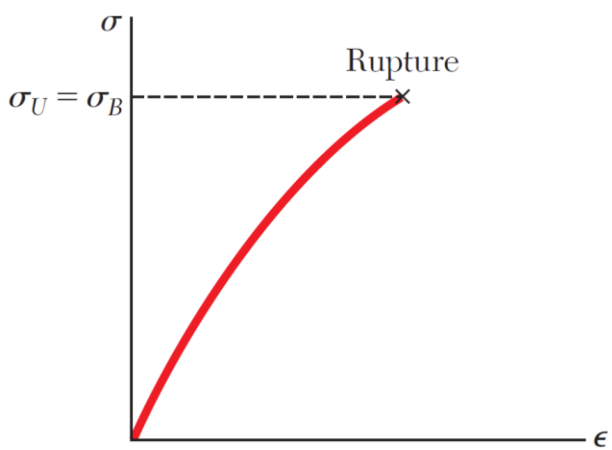


Fig. 2.7 Stress-strain diagram for a typical brittle material.

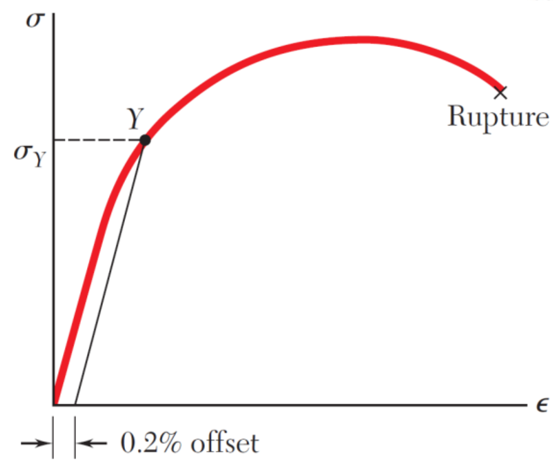


Fig. 2.8 Determination of yield strength by offset method.

Stress and Strain—Axial Loading

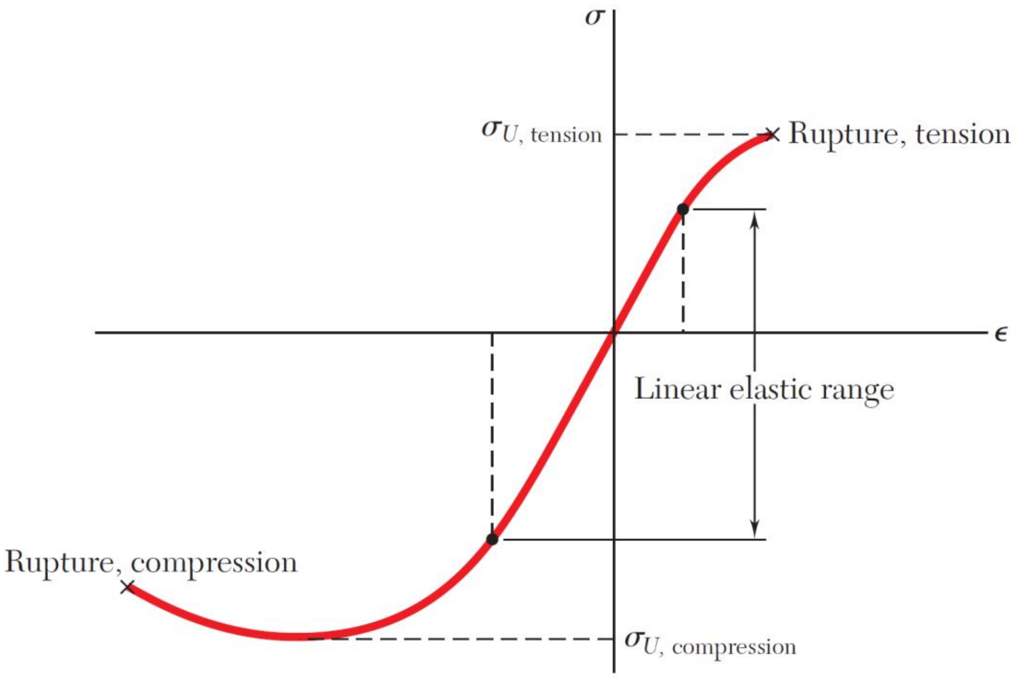


Fig. 2.9 Stress-strain diagram for concrete.

$$\epsilon_t = \Sigma\Delta\epsilon = \Sigma(\Delta L/L)$$

$$\epsilon_t = \int_{L_0}^{L} \frac{dL}{L} = \ln \frac{L}{L_0}$$

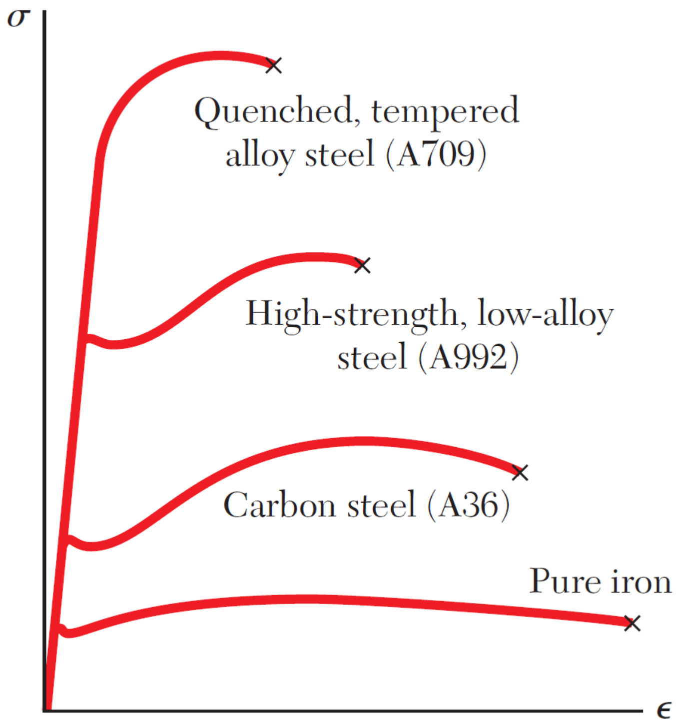


Fig. 2.11 Stress-strain diagrams for iron and different grades of steel.

$$\sigma = E\epsilon$$

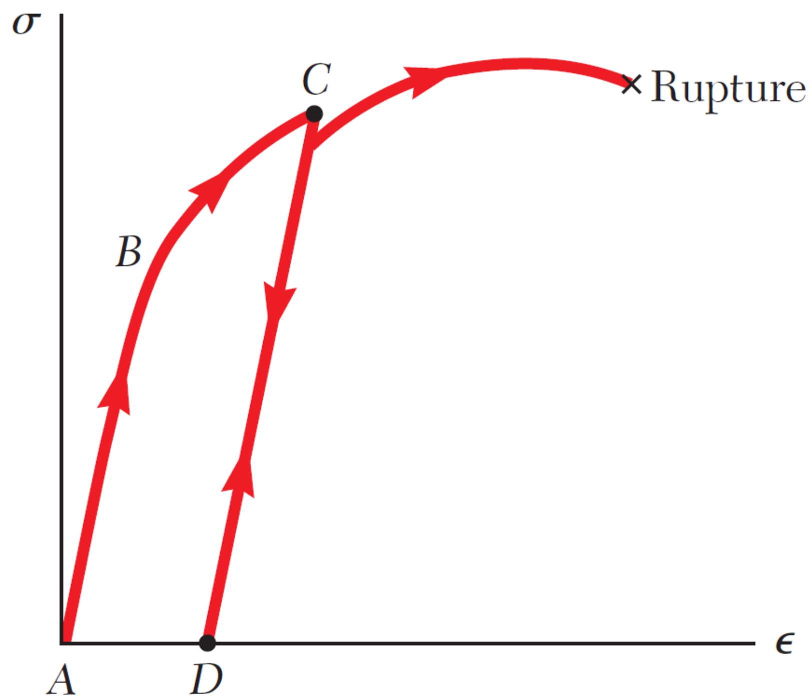


Fig. 2.14 Stress-strain characteristics of ductile material reloaded after prior yielding.

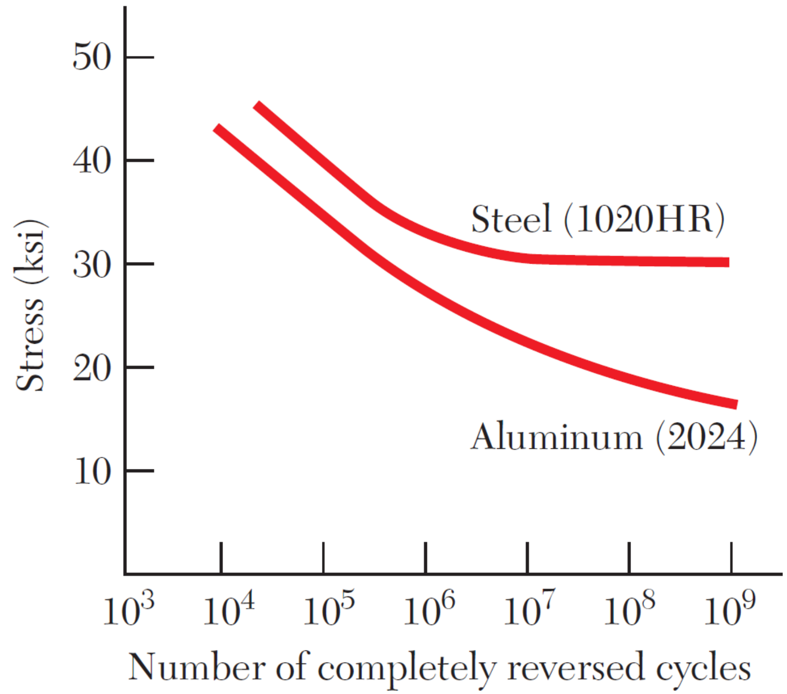


Fig. 2.16 Typical σ-n curves.

Consider a homogeneous rod BC of length L and uniform cross section of area A subjected to a centric axial load $\mathbf{P}$ (Fig. 2.17). If the resulting axial stress $\sigma = P/A$ does not exceed the proportional limit of the material, we may apply Hooke's law and write

$$\sigma = E\epsilon \tag{2.4}$$

from which it follows that

$$\epsilon = \frac{\sigma}{E} = \frac{P}{AE} \tag{2.5}$$

Recalling that the strain ϵ was defined in Sec. 2.2 as $\epsilon = \delta/L$, we have

$$\delta = \epsilon L \tag{2.6}$$

and, substituting for ϵ from (2.5) into (2.6):

$$\boxed{\delta = \frac{PL}{AE}}$$

$$\boxed{\delta = \sum_i \frac{P_i L_i}{A_i E_i}}$$

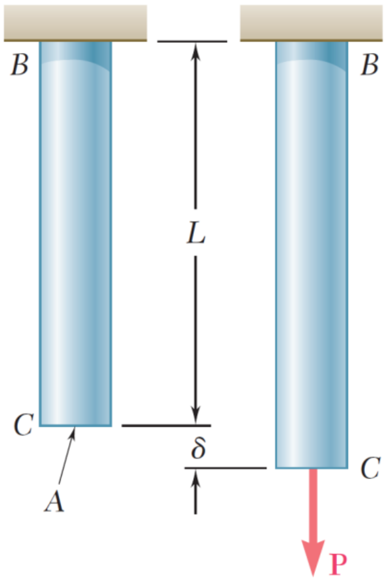


Fig. 2.17 Deformation of axially loaded rod.

$$d\delta = \epsilon \, dx = \frac{P \, dx}{AE}$$

$$\delta = \int_0^L \frac{P \, dx}{AE}$$

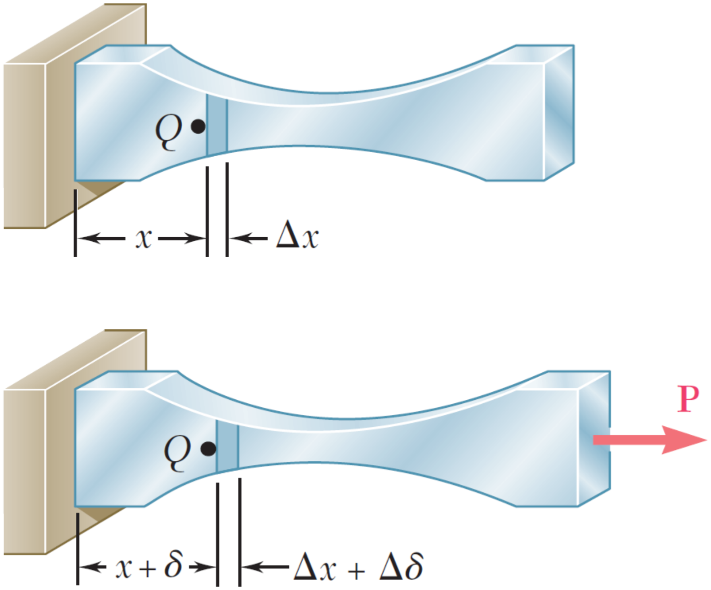
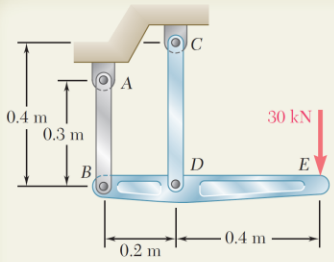
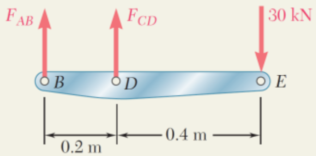


Fig. 2.18 Deformation of axially loaded member of variable cross-sectional area.



SAMPLE PROBLEM 2.1

The rigid bar *BDE* is supported by two links *AB* and *CD*. Link *AB* is made of aluminum (E = 70 GPa) and has a cross-sectional area of 500 mm^2; link *CD* is made of steel (E = 200 GPa) and has a cross-sectional area of 600 mm^2. For the 30-kN force shown, determine the deflection (*a*) of *B*, (*b*) of *D*, (*c*) of *E*.



SOLUTION

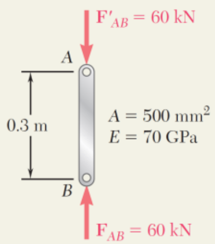
Free Body: Bar *BDE*

$$+\circlearrowleft\Sigma M_B = 0: \quad -(30 \text{ kN})(0.6 \text{ m}) + F_{CD}(0.2 \text{ m}) = 0$$

$$F_{CD} = +90 \text{ kN} \qquad F_{CD} = 90 \text{ kN} \quad \textit{tension}$$

$$+\circlearrowleft\Sigma M_D = 0: \quad -(30 \text{ kN})(0.4 \text{ m}) - F_{AB}(0.2 \text{ m}) = 0$$

$$F_{AB} = -60 \text{ kN} \qquad F_{AB} = 60 \text{ kN} \quad \textit{compression}$$

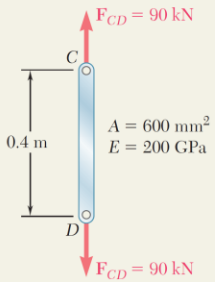


a. Deflection of *B*. Since the internal force in link *AB* is compressive, we have $P = -60$ kN

$$\delta_B = \frac{PL}{AE} = \frac{(-60 \times 10^3 \text{ N})(0.3 \text{ m})}{(500 \times 10^{-6} \text{ m}^2)(70 \times 10^9 \text{ Pa})} = -514 \times 10^{-6} \text{ m}$$

The negative sign indicates a contraction of member *AB*, and, thus, an upward deflection of end *B*:

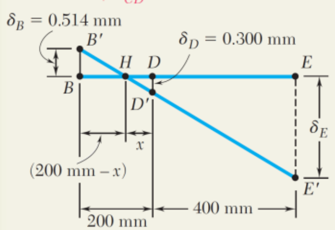
$$\delta_B = 0.514 \text{ mm} \uparrow \quad \blacktriangleleft$$



b. Deflection of *D*. Since in rod *CD*, P = 90 kN, we write

$$\delta_D = \frac{PL}{AE} = \frac{(90 \times 10^3 \text{ N})(0.4 \text{ m})}{(600 \times 10^{-6} \text{ m}^2)(200 \times 10^9 \text{ Pa})}$$

$$= 300 \times 10^{-6} \text{ m} \qquad \delta_D = 0.300 \text{ mm} \downarrow \quad \blacktriangleleft$$



c. Deflection of *E*. We denote by *B'* and *D'* the displaced positions of points *B* and *D*. Since the bar *BDE* is rigid, points *B'*, *D'*, and *E'* lie in a straight line and we write

$$\frac{BB'}{DD'} = \frac{BH}{HD} \qquad \frac{0.514 \text{ mm}}{0.300 \text{ mm}} = \frac{(200 \text{ mm}) - x}{x} \qquad x = 73.7 \text{ mm}$$

$$\frac{EE'}{DD'} = \frac{HE}{HD} \qquad \frac{\delta_E}{0.300 \text{ mm}} = \frac{(400 \text{ mm}) + (73.7 \text{ mm})}{73.7 \text{ mm}}$$

$$\delta_E = 1.928 \text{ mm} \downarrow \quad \blacktriangleleft$$

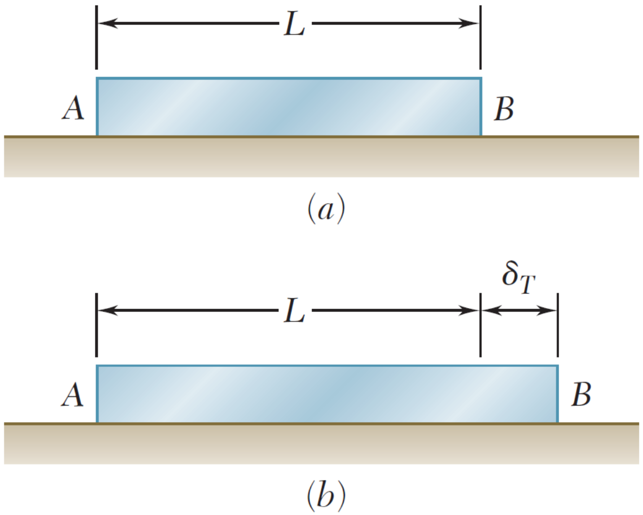


Fig. 2.30 Elongation of rod due to temperature increase.

$$\delta_T = \alpha(\Delta T)L$$

$$\epsilon_T = \alpha \Delta T$$

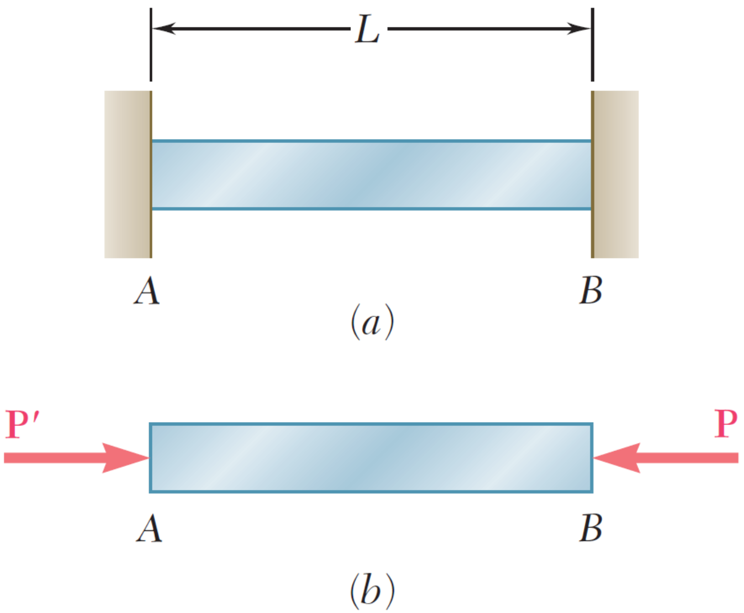


Fig. 2.31 Rod with ends restrained against thermal expansion.

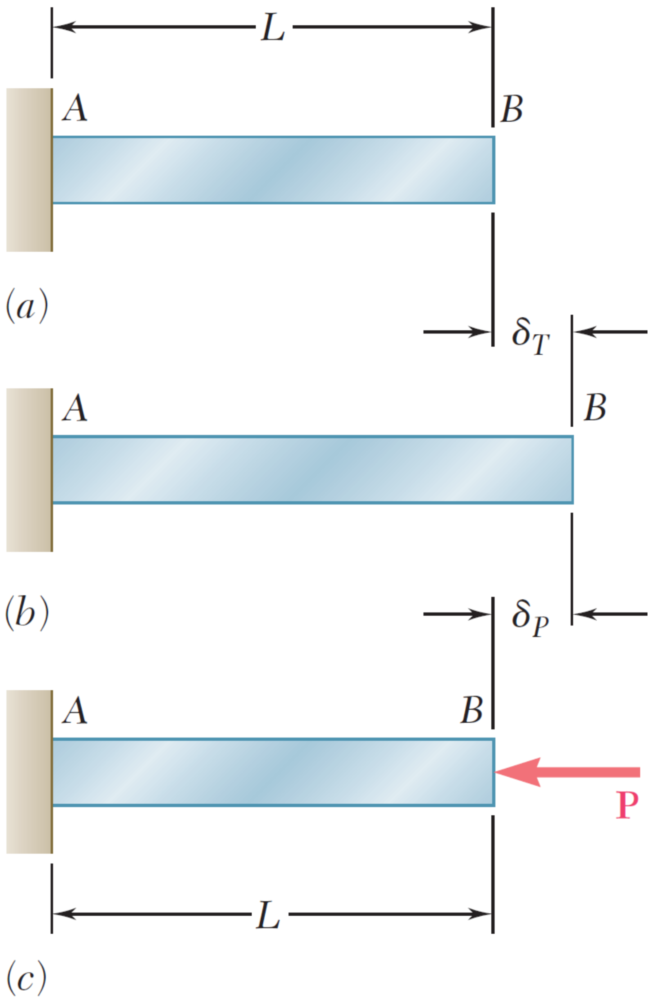


Fig. 2.32 Superposition method applied to rod restrained against thermal expansion.

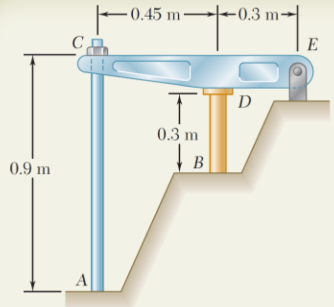
$$\delta_T = \alpha(\Delta T)L$$

$$\delta_P = \frac{PL}{AE}$$

$$\delta = \delta_T + \delta_P = \alpha(\Delta T)L + \frac{PL}{AE} = 0$$

$$P = -AE\alpha(\Delta T)$$

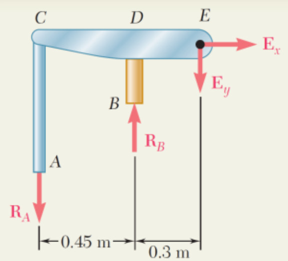
$$\sigma = \frac{P}{A} = -E\alpha(\Delta T)$$



SAMPLE PROBLEM 2.4

The rigid bar *CDE* is attached to a pin support at *E* and rests on the 30-mm-diameter brass cylinder *BD*. A 22-mm-diameter steel rod *AC* passes through a hole in the bar and is secured by a nut which is snugly fitted when the temperature of the entire assembly is 20°C. The temperature of the brass cylinder is then raised to 50°C while the steel rod remains at 20°C. Assuming that no stresses were present before the temperature change, determine the stress in the cylinder.

Rod *AC*: Steel	Cylinder *BD*: Brass
$E = 200$ GPa	$E = 105$ GPa
$\alpha = 11.7 \times 10^{-6}/°C$	$\alpha = 20.9 \times 10^{-6}/°C$



SOLUTION

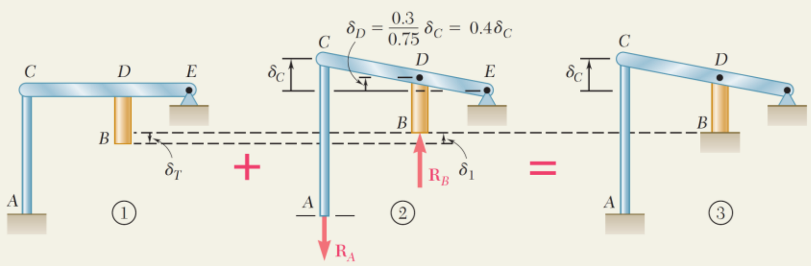
Statics. Considering the free body of the entire assembly, we write

$$+\curvearrowleft \Sigma M_E = 0: \qquad R_A(0.75 \text{ m}) - R_B(0.3 \text{ m}) = 0 \qquad R_A = 0.4R_B \tag{1}$$

Deformations. We use the method of superposition, considering $\mathbf{R}_B$ as redundant. With the support at *B* removed, the temperature rise of the cylinder causes point *B* to move down through δ_T. The reaction $\mathbf{R}_B$ must cause a deflection δ_1 equal to δ_T so that the final deflection of *B* will be zero (Fig. 3).

Deflection δ_T. Because of a temperature rise of $50° - 20° = 30°C$, the length of the brass cylinder increases by δ_T.

$$\delta_T = L(\Delta T)\alpha = (0.3 \text{ m})(30°C)(20.9 \times 10^{-6}/°C) = 188.1 \times 10^{-6} \text{ m} \downarrow$$



Deflection δ_1. We note that $\delta_D = 0.4\delta_C$ and $\delta_1 = \delta_D + \delta_{B/D}$.

$$\delta_C = \frac{R_A L}{AE} = \frac{R_A(0.9 \text{ m})}{\frac{1}{4}\pi(0.022 \text{ m})^2(200 \text{ GPa})} = 11.84 \times 10^{-9} R_A \uparrow$$

$$\delta_D = 0.40\delta_C = 0.4(11.84 \times 10^{-9} R_A) = 4.74 \times 10^{-9} R_A \uparrow$$

$$\delta_{B/D} = \frac{R_B L}{AE} = \frac{R_B(0.3 \text{ m})}{\frac{1}{4}\pi(0.03 \text{ m})^2(105 \text{ GPa})} = 4.04 \times 10^{-9} R_B \uparrow$$

We recall from (1) that $R_A = 0.4R_B$ and write

$$\delta_1 = \delta_D + \delta_{B/D} = [4.74(0.4R_B) + 4.04R_B]10^{-9} = 5.94 \times 10^{-9} R_B \uparrow$$

But $\delta_T = \delta_1$: $\qquad 188.1 \times 10^{-6} \text{ m} = 5.94 \times 10^{-9} R_B \qquad R_B = 31.7 \text{ kN}$

Stress in Cylinder: $\sigma_B = \dfrac{R_B}{A} = \dfrac{31.7 \text{ kN}}{\frac{1}{4}\pi(0.03 \text{ m})^2}$ $\qquad \sigma_B = 44.8$ MPa ◀

Stress and Strain—Axial Loading

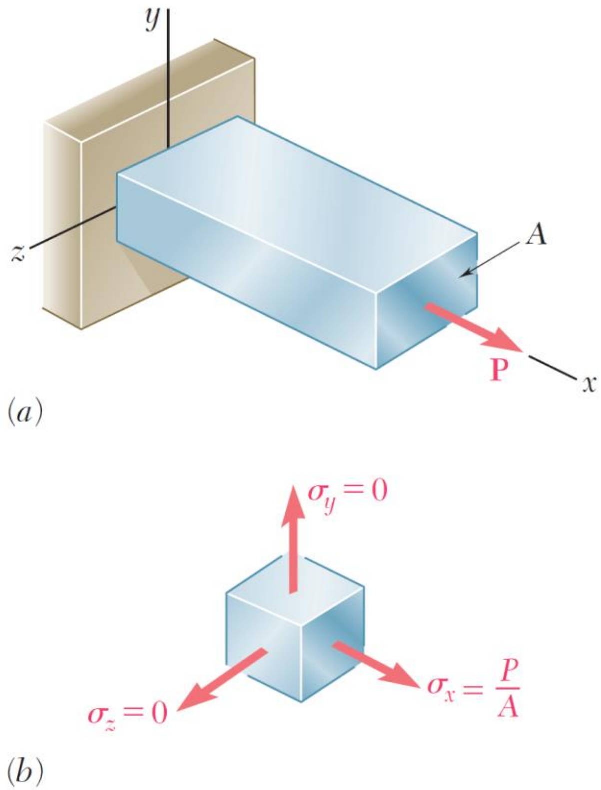


Fig. 2.35 Stresses in an axially-loaded bar.

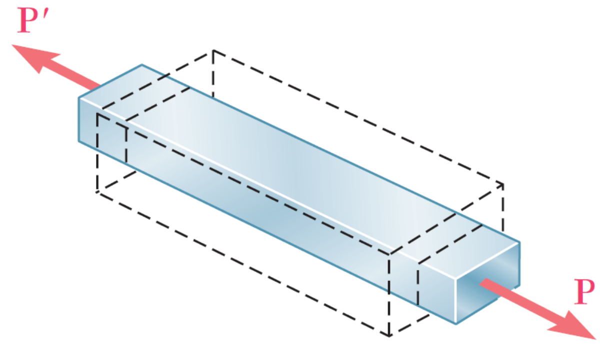


Fig. 2.36 Transverse contraction of bar under axial tensile force.

$$\nu = -\frac{\text{lateral strain}}{\text{axial strain}}$$

$$\nu = -\frac{\epsilon_y}{\epsilon_x} = -\frac{\epsilon_z}{\epsilon_x}$$

$$\epsilon_x = \frac{\sigma_x}{E}$$

$$\epsilon_y = \epsilon_z = -\frac{\nu\sigma_x}{E}$$

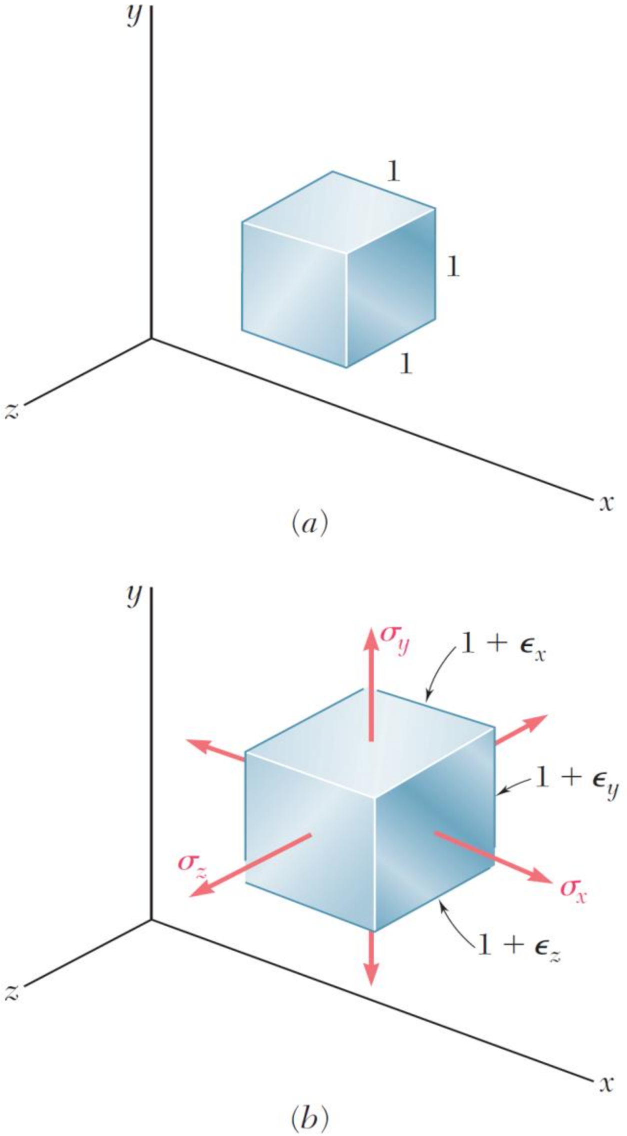


Fig. 2.39 Deformation of cube under multiaxial loading.

$$\epsilon_x = +\frac{\sigma_x}{E} - \frac{\nu\sigma_y}{E} - \frac{\nu\sigma_z}{E}$$
$$\epsilon_y = -\frac{\nu\sigma_x}{E} + \frac{\sigma_y}{E} - \frac{\nu\sigma_z}{E}$$
$$\epsilon_z = -\frac{\nu\sigma_x}{E} - \frac{\nu\sigma_y}{E} + \frac{\sigma_z}{E}$$

*2.13 DILATATION; BULK MODULUS

In this section you will examine the effect of the normal stresses σ_x, σ_y, and σ_z on the volume of an element of isotropic material. Consider the element shown in Fig. 2.39. In its unstressed state, it is in the shape of a cube of unit volume; and under the stresses σ_x, σ_y, σ_z, it deforms into a rectangular parallelepiped of volume

$$v = (1 + \epsilon_x)(1 + \epsilon_y)(1 + \epsilon_z)$$

Since the strains ϵ_x, ϵ_y, ϵ_z are much smaller than unity, their products will be even smaller and may be omitted in the expansion of the product. We have, therefore,

$$v = 1 + \epsilon_x + \epsilon_y + \epsilon_z$$

Denoting by e the change in volume of our element, we write

$$e = v - 1 = 1 + \epsilon_x + \epsilon_y + \epsilon_z - 1$$

or

$$e = \epsilon_x + \epsilon_y + \epsilon_z \qquad (2.30)$$

Since the element had originally a unit volume, the quantity e represents *the change in volume per unit volume;* it is referred to as the *dilatation* of the material. Substituting for ϵ_x, ϵ_y, and ϵ_z from Eqs. (2.28) into (2.30), we write

$$e = \frac{\sigma_x + \sigma_y + \sigma_z}{E} - \frac{2\nu(\sigma_x + \sigma_y + \sigma_z)}{E}$$

$$e = \frac{1 - 2\nu}{E}(\sigma_x + \sigma_y + \sigma_z) \qquad (2.31)\dagger$$

A case of special interest is that of a body subjected to a uniform hydrostatic pressure p. Each of the stress components is then equal to $-p$ and Eq. (2.31) yields

$$e = -\frac{3(1 - 2\nu)}{E}p \qquad (2.32)$$

Introducing the constant

$$k = \frac{E}{3(1 - 2\nu)} \qquad (2.33)$$

we write Eq. (2.32) in the form

$$e = -\frac{p}{k} \qquad (2.34)$$

Stress and Strain—Axial Loading

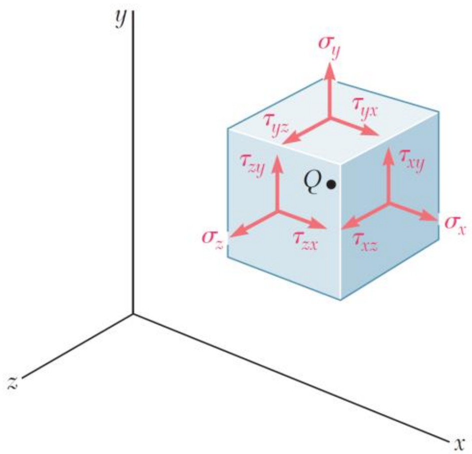


Fig. 2.41 General state of stress.

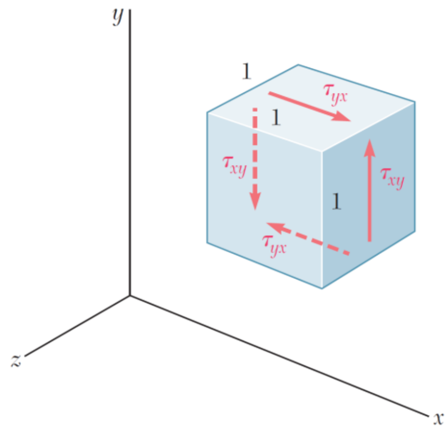


Fig. 2.42 Cubic element subjected to shearing stresses.

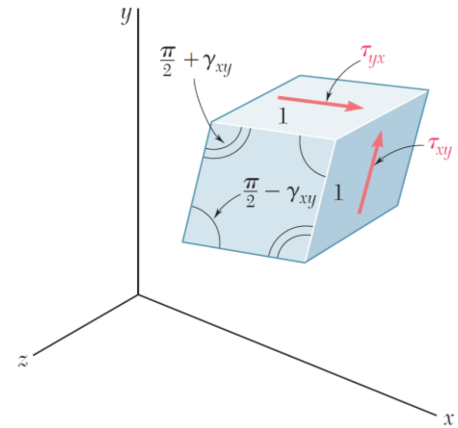


Fig. 2.43 Deformation of cubic element due to shearing stresses.

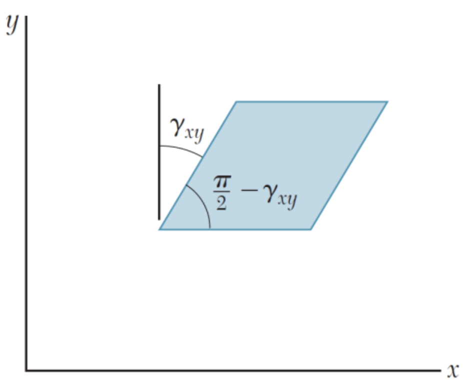


Fig. 2.44

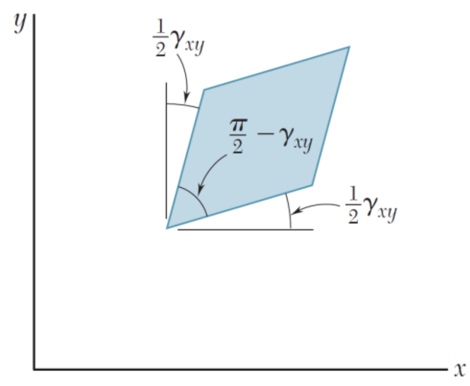


Fig. 2.45

$$\tau_{xy} = G\gamma_{xy}$$

$$\tau_{yz} = G\gamma_{yz}$$

$$\tau_{zx} = G\gamma_{zx}$$

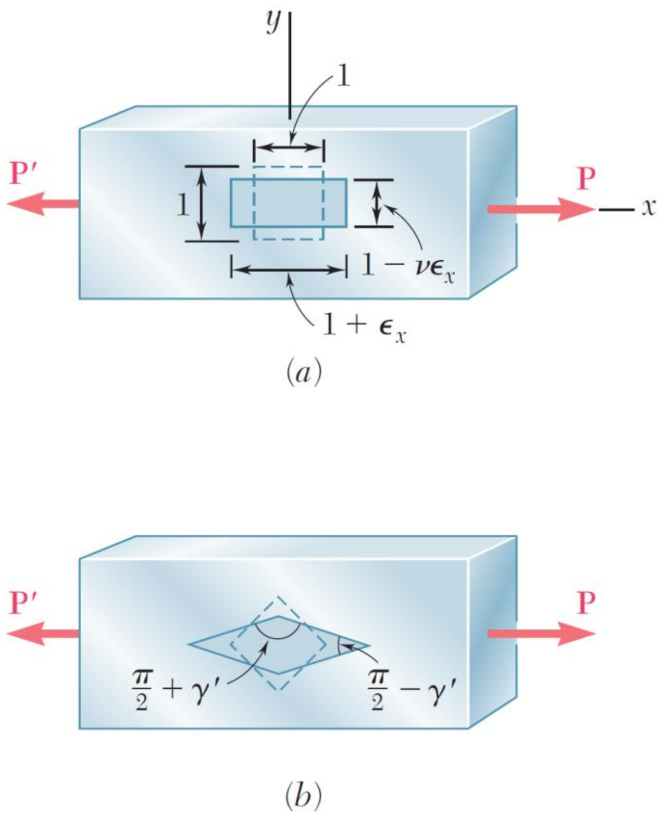


Fig. 2.49 Representations of strain in an axially-loaded bar.

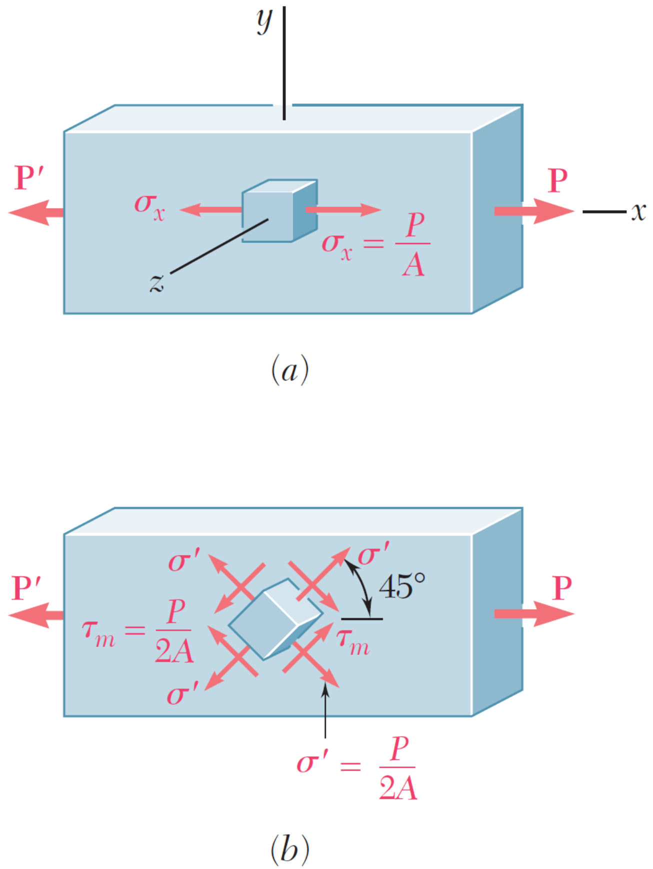


Fig. 1.38 *(repeated)*

$$\beta = \frac{\pi}{4} - \frac{\gamma_m}{2}$$

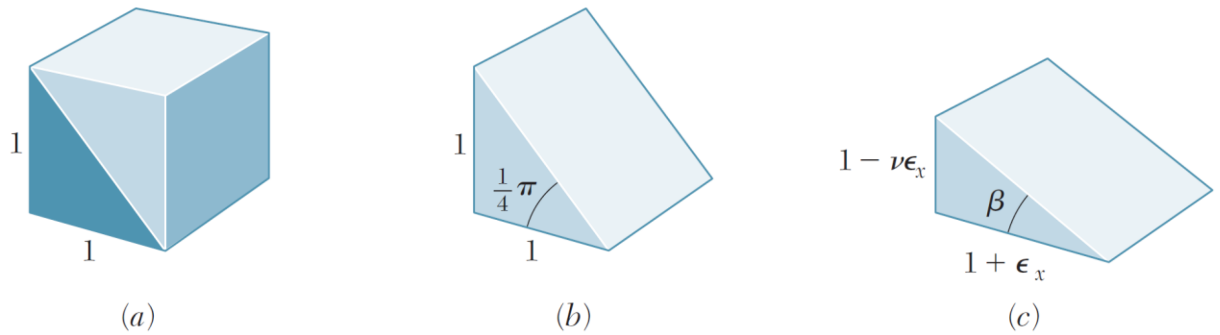


Fig. 2.50

Applying the formula for the tangent of the difference of two angles, we obtain

$$\tan \beta = \frac{\tan \dfrac{\pi}{4} - \tan \dfrac{\gamma_m}{2}}{1 + \tan \dfrac{\pi}{4} \tan \dfrac{\gamma_m}{2}} = \frac{1 - \tan \dfrac{\gamma_m}{2}}{1 + \tan \dfrac{\gamma_m}{2}}$$

or, since $\gamma_m/2$ is a very small angle,

$$\tan \beta = \frac{1 - \dfrac{\gamma_m}{2}}{1 + \dfrac{\gamma_m}{2}} \qquad (2.39)$$

But, from Fig. 2.50*c*, we observe that

$$\tan \beta = \frac{1 - \nu\epsilon_x}{1 + \epsilon_x} \tag{2.40}$$

Equating the right-hand members of (2.39) and (2.40), and solving for γ_m, we write

$$\gamma_m = \frac{(1 + \nu)\epsilon_x}{1 + \dfrac{1 - \nu}{2}\epsilon_x}$$

Since $\epsilon_x \ll 1$, the denominator in the expression obtained can be assumed equal to one; we have, therefore,

$$\gamma_m = (1 + \nu)\epsilon_x \tag{2.41}$$

$$\boxed{\gamma_m = (1 + \nu)\epsilon_x}$$

$$\gamma_m = (1 + \nu)\epsilon_x$$

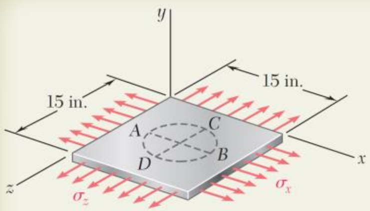
$$\frac{\tau_m}{G} = (1 + \nu)\frac{\sigma_x}{E}$$

$$\frac{E}{G} = (1 + \nu)\frac{\sigma_x}{\tau_m}$$

$\sigma_x = P/A$ and $\tau_m = P/2A$

$$\frac{E}{2G} = 1 + \nu$$

$$G = \frac{E}{2(1 + \nu)}$$



SAMPLE PROBLEM 2.5

A circle of diameter $d = 9$ in. is scribed on an unstressed aluminum plate of thickness $t = \frac{3}{4}$ in. Forces acting in the plane of the plate later cause normal stresses $\sigma_x = 12$ ksi and $\sigma_z = 20$ ksi. For $E = 10 \times 10^6$ psi and $\nu = \frac{1}{3}$, determine the change in (*a*) the length of diameter *AB*, (*b*) the length of diameter *CD*, (*c*) the thickness of the plate, (*d*) the volume of the plate.

SOLUTION

Hooke's Law. We note that $\sigma_y = 0$. Using Eqs. (2.28) we find the strain in each of the coordinate directions.

$$\epsilon_x = +\frac{\sigma_x}{E} - \frac{\nu\sigma_y}{E} - \frac{\nu\sigma_z}{E}$$

$$= \frac{1}{10 \times 10^6 \text{ psi}}\left[(12 \text{ ksi}) - 0 - \frac{1}{3}(20 \text{ ksi})\right] = +0.533 \times 10^{-3} \text{ in./in.}$$

$$\epsilon_y = -\frac{\nu\sigma_x}{E} + \frac{\sigma_y}{E} - \frac{\nu\sigma_z}{E}$$

$$= \frac{1}{10 \times 10^6 \text{ psi}}\left[-\frac{1}{3}(12 \text{ ksi}) + 0 - \frac{1}{3}(20 \text{ ksi})\right] = -1.067 \times 10^{-3} \text{ in./in.}$$

$$\epsilon_z = -\frac{\nu\sigma_x}{E} - \frac{\nu\sigma_y}{E} + \frac{\sigma_z}{E}$$

$$= \frac{1}{10 \times 10^6 \text{ psi}}\left[-\frac{1}{3}(12 \text{ ksi}) - 0 + (20 \text{ ksi})\right] = +1.600 \times 10^{-3} \text{ in./in.}$$

a. Diameter *AB*. The change in length is $\delta_{B/A} = \epsilon_x d$.

$$\delta_{B/A} = \epsilon_x d = (+0.533 \times 10^{-3} \text{ in./in.})(9 \text{ in.})$$

$$\delta_{B/A} = +4.8 \times 10^{-3} \text{ in.} \quad ◀$$

b. Diameter *CD*.

$$\delta_{C/D} = \epsilon_z d = (+1.600 \times 10^{-3} \text{ in./in.})(9 \text{ in.})$$

$$\delta_{C/D} = +14.4 \times 10^{-3} \text{ in.} \quad ◀$$

c. Thickness. Recalling that $t = \frac{3}{4}$ in., we have

$$\delta_t = \epsilon_y t = (-1.067 \times 10^{-3} \text{ in./in.})(\tfrac{3}{4} \text{ in.})$$

$$\delta_t = -0.800 \times 10^{-3} \text{ in.} \quad ◀$$

d. Volume of the Plate. Using Eq. (2.30), we write

$$e = \epsilon_x + \epsilon_y + \epsilon_z = (+0.533 - 1.067 + 1.600)10^{-3} = +1.067 \times 10^{-3}$$

$$\Delta V = eV = +1.067 \times 10^{-3}[(15 \text{ in.})(15 \text{ in.})(\tfrac{3}{4} \text{ in.})] \quad \Delta V = +0.187 \times \text{in}^3 \quad ◀$$

STRESS CONCENTRATIONS

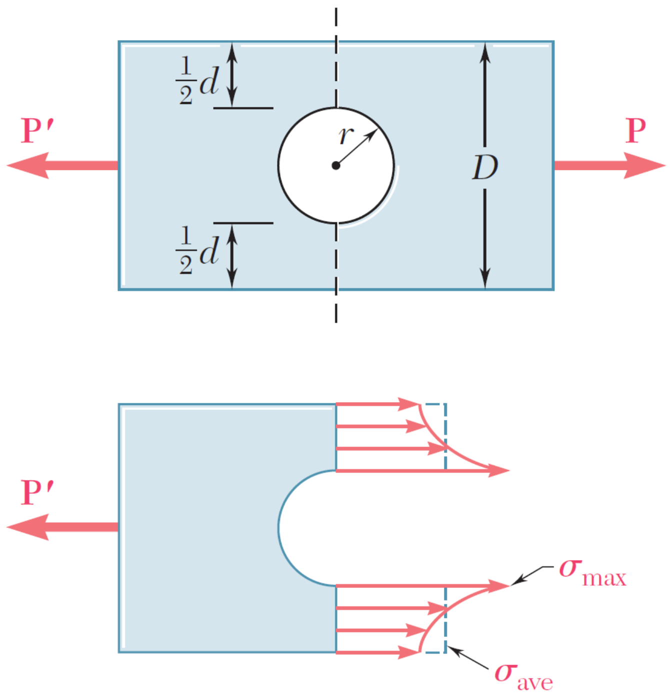


Fig. 2.58 Stress distribution near circular hole in flat bar under axial loading.

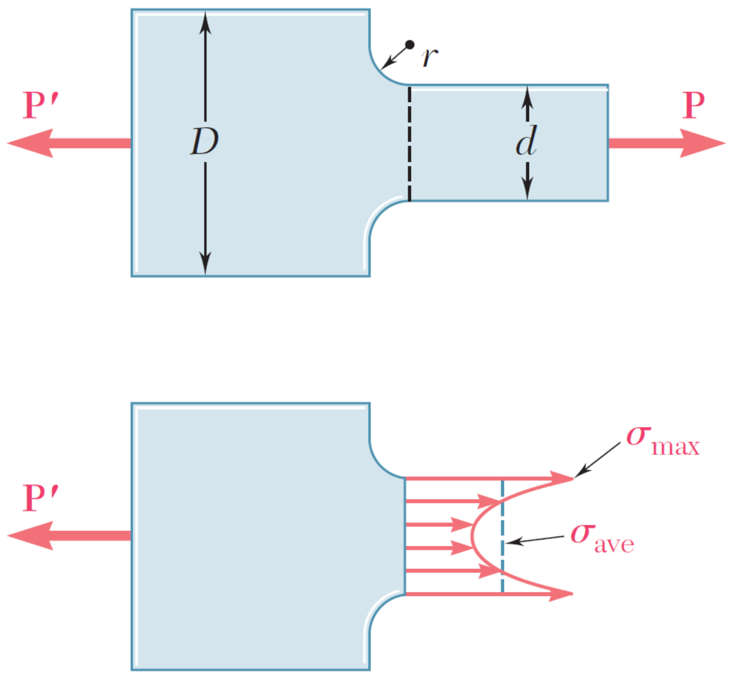


Fig. 2.59 Stress distribution near fillets in flat bar under axial loading.

$$K = \frac{\sigma_{max}}{\sigma_{ave}}$$

Torsion

Torsion

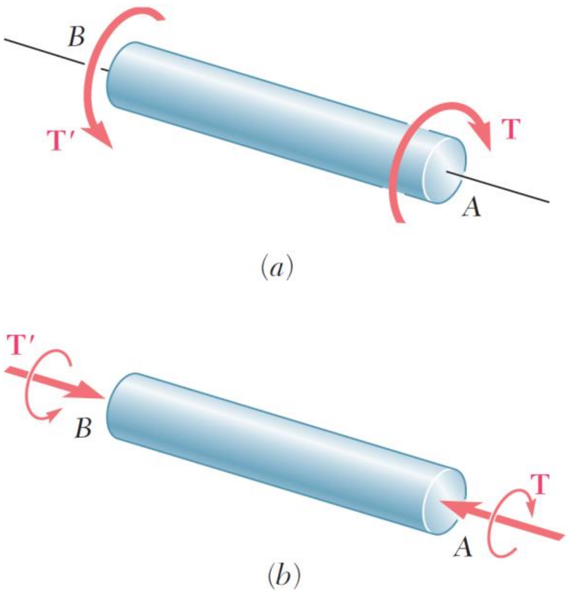


Fig. 3.1 Shaft subject to torsion.

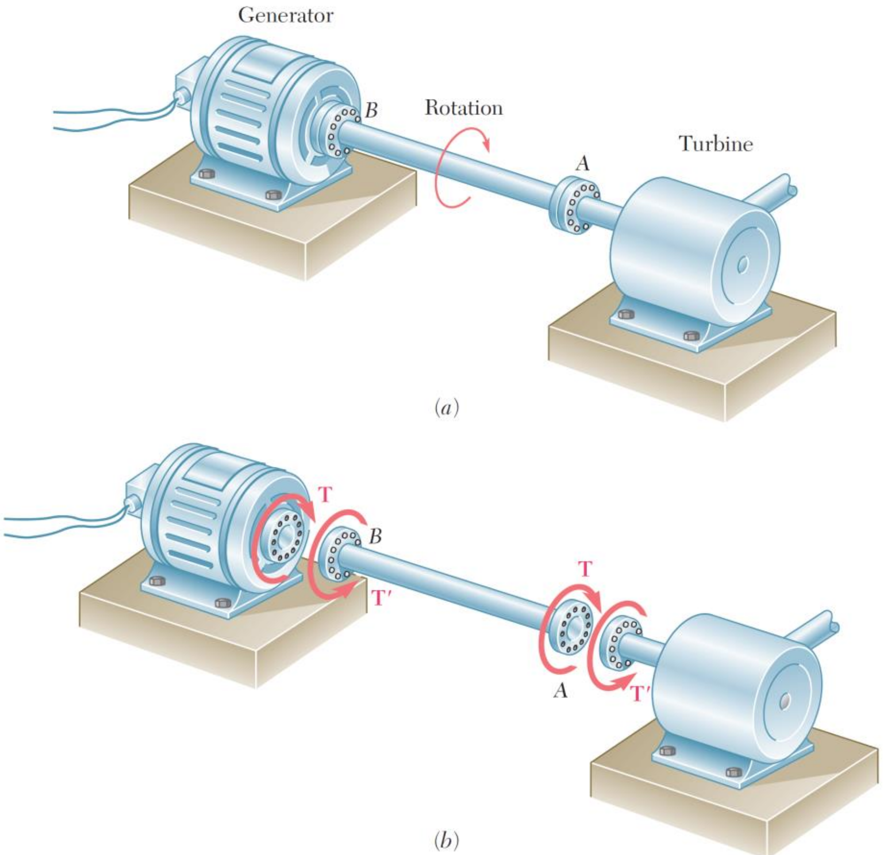
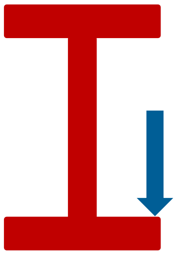


Fig. 3.2 Transmission shaft.



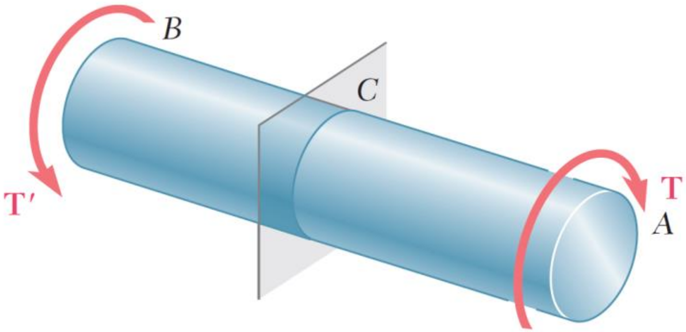


Fig. 3.3 Shaft subject to torques.

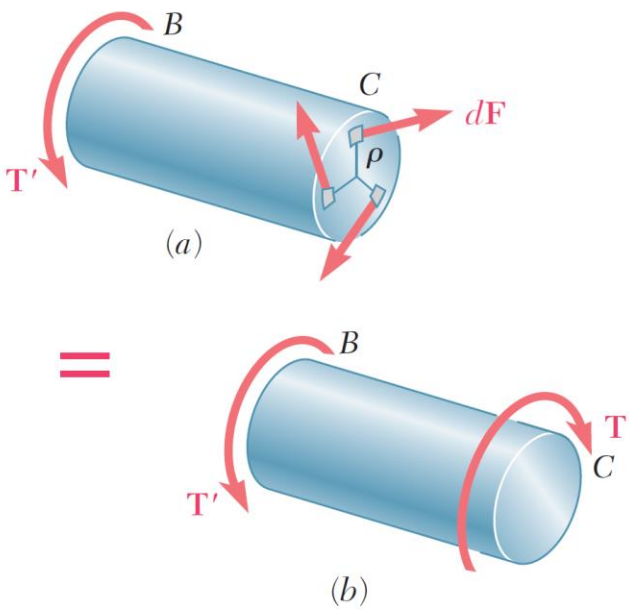


Fig. 3.4

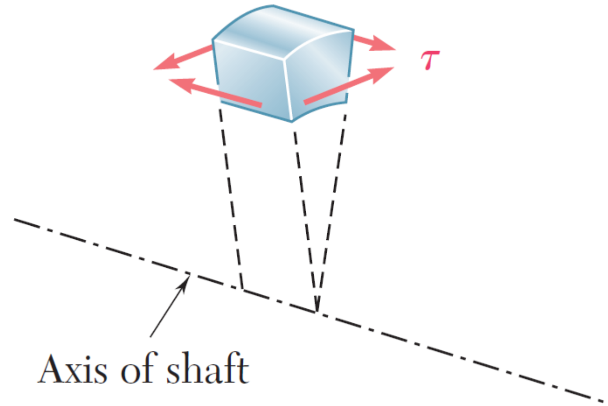


Fig. 3.5 Element in shaft.

$$\int \rho \, dF = T$$

$$\int \rho (\tau \, dA) = T$$

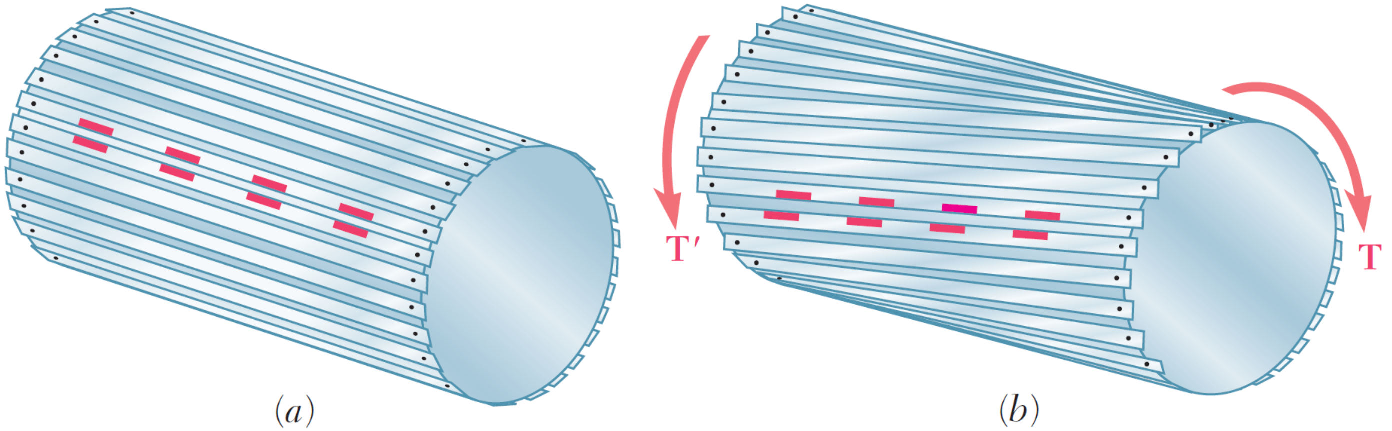


Fig. 3.6 Model of shaft.

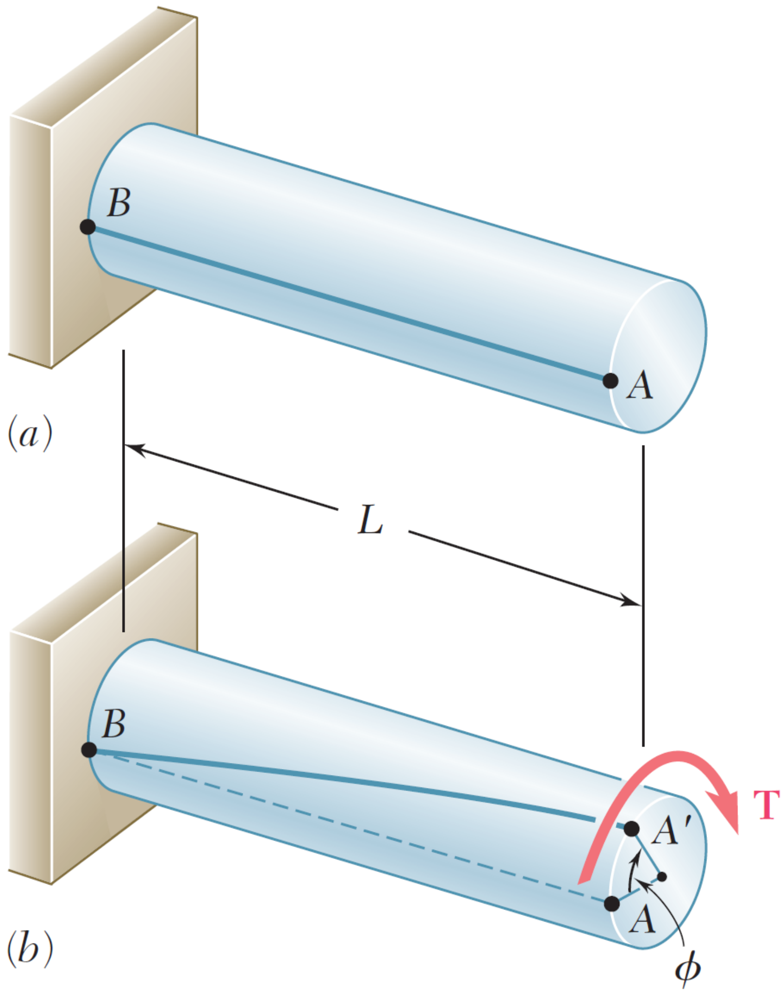


Fig. 3.7 Shaft with fixed support.

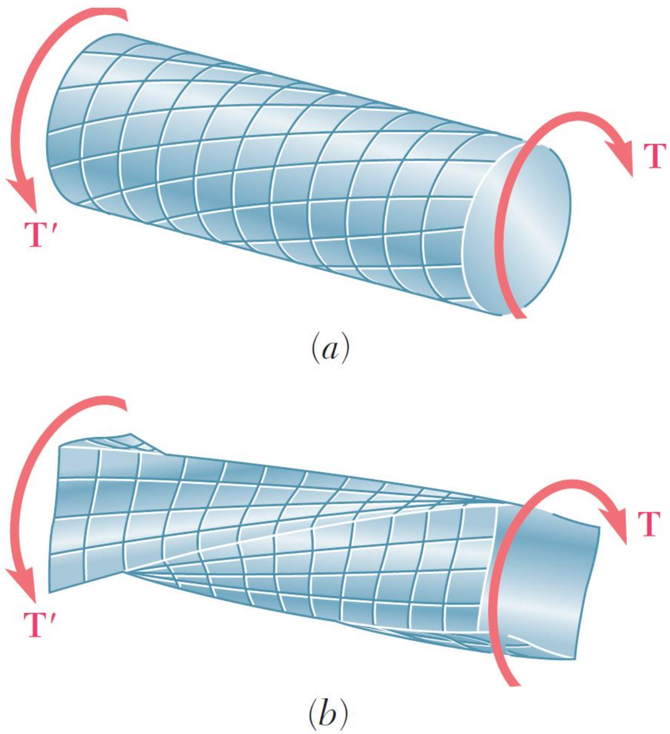


Fig. 3.8 Comparison of deformations in circular and square shafts.

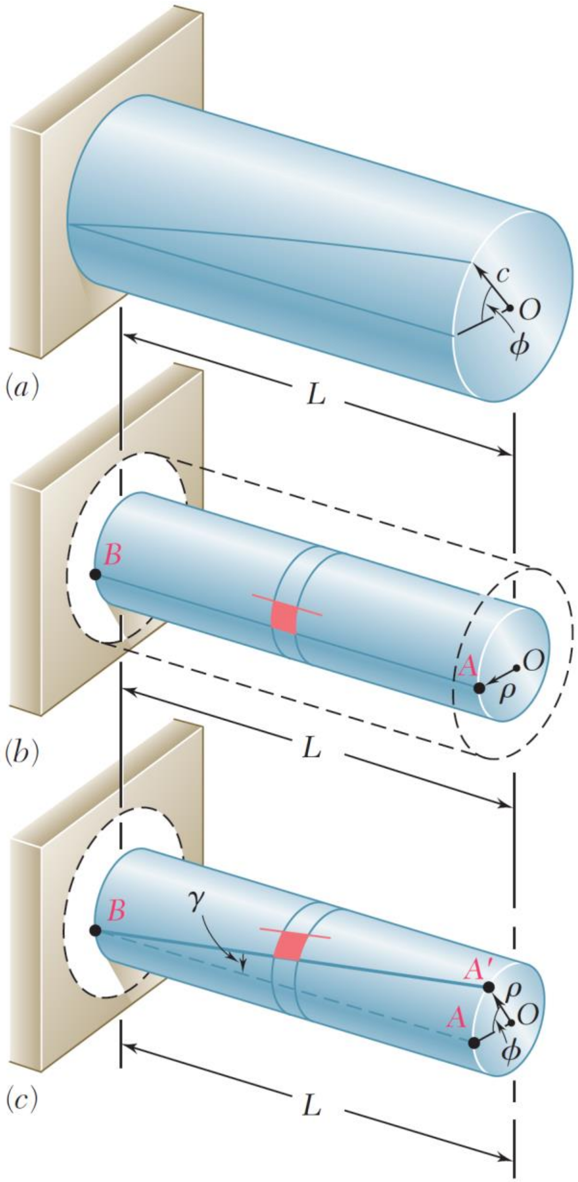


Fig. 3.13 Shearing strain.

We observe from Fig. 3.13*c* that, for small values of γ, we can express the arc length AA' as $AA' = L\gamma$. But, on the other hand, we have $AA' = \rho\phi$. It follows that $L\gamma = \rho\phi$, or

$$\gamma = \frac{\rho\phi}{L} \tag{3.2}$$

It follows from Eq. (3.2) that the shearing strain is maximum on the surface of the shaft, where $\rho = c$. We have

$$\gamma_{\max} = \frac{c\phi}{L} \tag{3.3}$$

Eliminating ϕ from Eqs. (3.2) and (3.3), we can express the shearing strain γ at a distance ρ from the axis of the shaft as

$$\gamma = \frac{\rho}{c}\gamma_{\max} \tag{3.4}$$

Recalling Hooke's law for shearing stress and strain from Sec. 2.14, we write

$$\tau = G\gamma \tag{3.5}$$

where G is the modulus of rigidity or shear modulus of the material. Multiplying both members of Eq. (3.4) by G, we write

$$G\gamma = \frac{\rho}{c} G\gamma_{\max}$$

or, making use of Eq. (3.5),

$$\boxed{\tau = \frac{\rho}{c} \tau_{\max}} \tag{3.6}$$

shaft of inner radius c_1 and outer radius c_2. From Eq. (3.6), we find that, in the latter case,

$$\tau_{\min} = \frac{c_1}{c_2} \tau_{\max} \tag{3.7}$$

We now recall from Sec. 3.2 that the sum of the moments of the elementary forces exerted on any cross section of the shaft must be equal to the magnitude T of the torque exerted on the shaft:

$$\int \rho(\tau \, dA) = T \tag{3.1}$$

Substituting for τ from (3.6) into (3.1), we write

$$T = \int \rho\tau\, dA = \frac{\tau_{\max}}{c}\int \rho^2\, dA$$

But the integral in the last member represents the polar moment of inertia J of the cross section with respect to its center O. We have therefore

$$T = \frac{\tau_{\max} J}{c} \tag{3.8}$$

or, solving for $\tau_{\max}$,

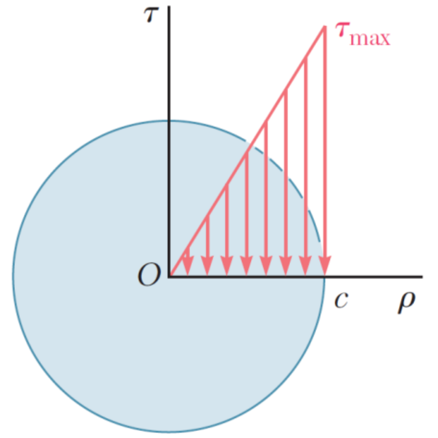
$$\boxed{\tau_{\max} = \frac{Tc}{J}} \tag{3.9}$$

Substituting for $\tau_{\max}$ from (3.9) into (3.6), we express the shearing stress at any distance ρ from the axis of the shaft as

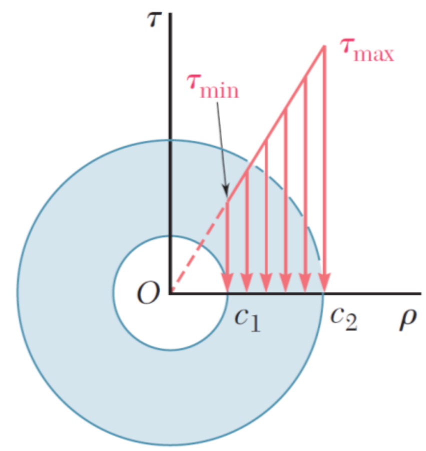
$$\boxed{\tau = \frac{T\rho}{J}} \tag{3.10}$$

Equations (3.9) and (3.10) are known as the *elastic torsion formulas*. We recall from statics that the polar moment of inertia of a circle of radius c is $J = \frac{1}{2}\pi c^4$. In the case of a hollow circular shaft of inner radius c_1 and outer radius c_2, the polar moment of inertia is

$$J = \tfrac{1}{2}\pi c_2^4 - \tfrac{1}{2}\pi c_1^4 = \tfrac{1}{2}\pi(c_2^4 - c_1^4) \tag{3.11}$$



(a)



(b)

$$F = 2(\tau_{max}A_0)\cos 45° = \tau_{max}A_0\sqrt{2} \qquad (3.13)$$

The corresponding stress is obtained by dividing the force F by the area A of face DC. Observing that $A = A_0\sqrt{2}$, we write

$$\sigma = \frac{F}{A} = \frac{\tau_{max}A_0\sqrt{2}}{A_0\sqrt{2}} = \tau_{max} \qquad (3.14)$$

A similar analysis of the element of Fig. 3.18b shows that the stress on the face BE is $\sigma = -\tau_{max}$. We conclude that the stresses exerted on the faces of an element c at 45° to the axis of the shaft (Fig. 3.19) are normal stresses equal to $\pm\tau_{max}$. Thus, while the element a in Fig. 3.19 is in pure shear, the element c in the same figure is subjected to a tensile stress on two of its faces, and to a compressive stress on the other two. We also note that all the stresses involved have the same magnitude, Tc/J.†

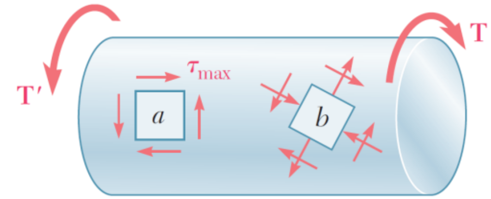


Fig. 3.17 Circular shaft with elements at different orientations.

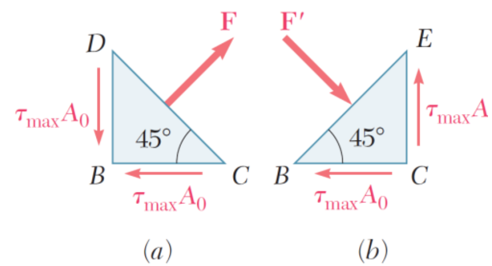


Fig. 3.18 Forces on faces at 45° to shaft axis.

Photo 3.2 Shear failure of shaft subject to torque.

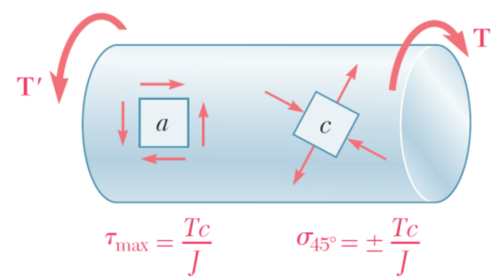
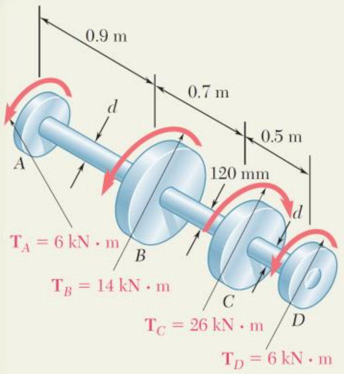


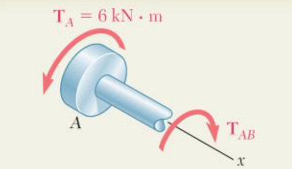
Fig. 3.19 Shaft with elements with only shear stresses or normal stresses.

SAMPLE PROBLEM 3.1



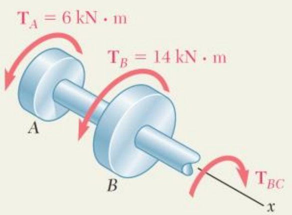
Shaft BC is hollow with inner and outer diameters of 90 mm and 120 mm, respectively. Shafts AB and CD are solid and of diameter d. For the loading shown, determine (*a*) the maximum and minimum shearing stress in shaft BC, (*b*) the required diameter d of shafts AB and CD if the allowable shearing stress in these shafts is 65 MPa.

SOLUTION



Equations of Statics. Denoting by $\mathbf{T}_{AB}$ the torque in shaft AB, we pass a section through shaft AB and, for the free body shown, we write

$$\Sigma M_x = 0: \qquad (6 \text{ kN} \cdot \text{m}) - T_{AB} = 0 \qquad T_{AB} = 6 \text{ kN} \cdot \text{m}$$

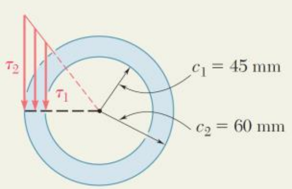


We now pass a section through shaft BC and, for the free body shown, we have

$$\Sigma M_x = 0: \quad (6 \text{ kN} \cdot \text{m}) + (14 \text{ kN} \cdot \text{m}) - T_{BC} = 0 \qquad T_{BC} = 20 \text{ kN} \cdot \text{m}$$

a. Shaft *BC*. For this hollow shaft we have

$$J = \frac{\pi}{2}(c_2^4 - c_1^4) = \frac{\pi}{2}[(0.060)^4 - (0.045)^4] = 13.92 \times 10^{-6} \text{ m}^4$$

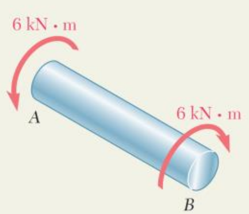


Maximum Shearing Stress. On the outer surface, we have

$$\tau_{\max} = \tau_2 = \frac{T_{BC}c_2}{J} = \frac{(20 \text{ kN} \cdot \text{m})(0.060 \text{ m})}{13.92 \times 10^{-6} \text{ m}^4} \qquad \tau_{\max} = 86.2 \text{ MPa} \blacktriangleleft$$

Minimum Shearing Stress. We write that the stresses are proportional to the distance from the axis of the shaft.

$$\frac{\tau_{\min}}{\tau_{\max}} = \frac{c_1}{c_2} \qquad \frac{\tau_{\min}}{86.2 \text{ MPa}} = \frac{45 \text{ mm}}{60 \text{ mm}} \qquad \tau_{\min} = 64.7 \text{ MPa} \blacktriangleleft$$



b. Shafts *AB* and *CD*. We note that in both of these shafts the magnitude of the torque is $T = 6$ kN · m and $\tau_{\text{all}} = 65$ MPa. Denoting by c the radius of the shafts, we write

$$\tau = \frac{Tc}{J} \qquad 65 \text{ MPa} = \frac{(6 \text{ kN} \cdot \text{m})c}{\frac{\pi}{2}c^4}$$

$$c^3 = 58.8 \times 10^{-6} \text{ m}^3 \qquad c = 38.9 \times 10^{-3} \text{ m}$$

$$d = 2c = 2(38.9 \text{ mm}) \qquad d = 77.8 \text{ mm} \blacktriangleleft$$

3.5 ANGLE OF TWIST IN THE ELASTIC RANGE

In this section, a relation will be derived between the angle of twist ϕ of a circular shaft and the torque **T** exerted on the shaft. The entire shaft will be assumed to remain elastic. Considering first the case of a shaft of length L and of uniform cross section of radius c subjected to a torque **T** at its free end (Fig. 3.20), we recall from Sec. 3.3 that the angle of twist ϕ and the maximum shearing strain γ_{max} are related as follows:

$$\gamma_{max} = \frac{c\phi}{L} \qquad (3.3)$$

But, in the elastic range, the yield stress is not exceeded anywhere in the shaft, Hooke's law applies, and we have $\gamma_{max} = \tau_{max}/G$ or, recalling Eq. (3.9),

$$\gamma_{max} = \frac{\tau_{max}}{G} = \frac{Tc}{JG} \qquad (3.15)$$

Equating the right-hand members of Eqs. (3.3) and (3.15), and solving for ϕ, we write

$$\boxed{\phi = \frac{TL}{JG}} \qquad (3.16)$$

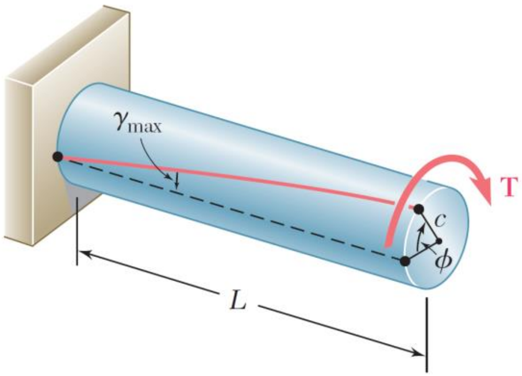


Fig. 3.20 Angle of twist ϕ.

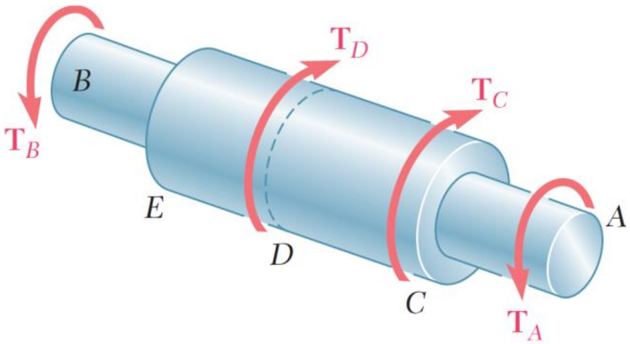


Fig. 3.21 Multiple sections and multiple torques.

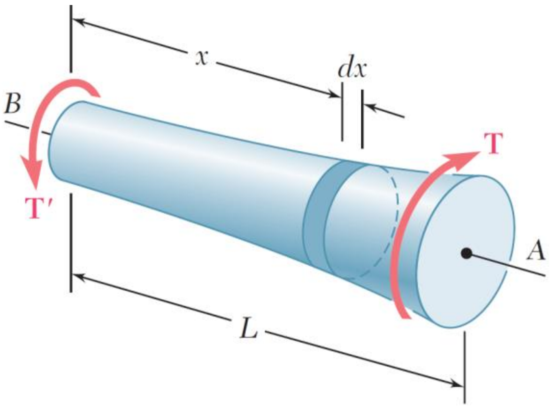
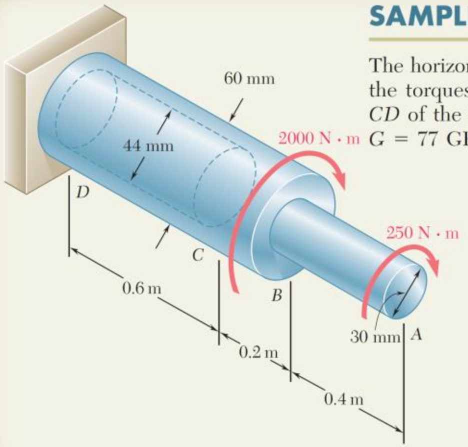


Fig. 3.22 Shaft with variable cross section.

$$\phi = \sum_i \frac{T_i L_i}{J_i G_i}$$

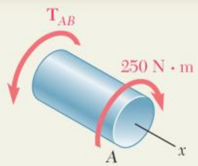
$$\phi = \int_0^L \frac{T\,dx}{JG}$$

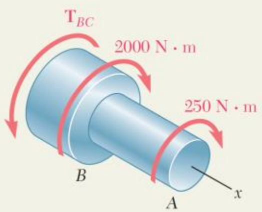
SAMPLE PROBLEM 3.3

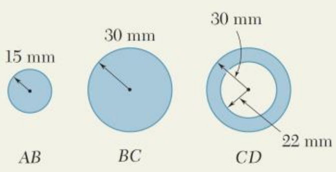


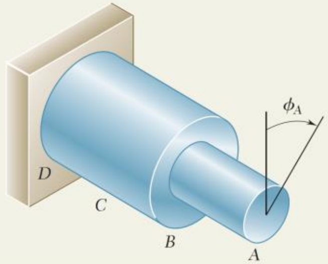
The horizontal shaft AD is attached to a fixed base at D and is subjected to the torques shown. A 44-mm-diameter hole has been drilled into portion CD of the shaft. Knowing that the entire shaft is made of steel for which $G = 77$ GPa, determine the angle of twist at end A.

SOLUTION









Since the shaft consists of three portions AB, BC, and CD, each of uniform cross section and each with a constant internal torque, Eq. (3.17) may be used.

Statics. Passing a section through the shaft between A and B and using the free body shown, we find

$$\Sigma M_x = 0: \qquad (250\ \text{N}\cdot\text{m}) - T_{AB} = 0 \qquad T_{AB} = 250\ \text{N}\cdot\text{m}$$

Passing now a section between B and C, we have

$$\Sigma M_x = 0: (250\ \text{N}\cdot\text{m}) + (2000\ \text{N}\cdot\text{m}) - T_{BC} = 0 \qquad T_{BC} = 2250\ \text{N}\cdot\text{m}$$

Since no torque is applied at C,

$$T_{CD} = T_{BC} = 2250\ \text{N}\cdot\text{m}$$

Polar Moments of Inertia

$$J_{AB} = \frac{\pi}{2}c^4 = \frac{\pi}{2}(0.015\ \text{m})^4 = 0.0795 \times 10^{-6}\ \text{m}^4$$

$$J_{BC} = \frac{\pi}{2}c^4 = \frac{\pi}{2}(0.030\ \text{m})^4 = 1.272 \times 10^{-6}\ \text{m}^4$$

$$J_{CD} = \frac{\pi}{2}(c_2^4 - c_1^4) = \frac{\pi}{2}[(0.030\ \text{m})^4 - (0.022\ \text{m})^4] = 0.904 \times 10^{-6}\ \text{m}^4$$

Angle of Twist. Using Eq. (3.17) and recalling that $G = 77$ GPa for the entire shaft, we have

$$\phi_A = \sum_i \frac{T_i L_i}{J_i G} = \frac{1}{G}\left(\frac{T_{AB}L_{AB}}{J_{AB}} + \frac{T_{BC}L_{BC}}{J_{BC}} + \frac{T_{CD}L_{CD}}{J_{CD}}\right)$$

$$\phi_A = \frac{1}{77\ \text{GPa}}\left[\frac{(250\ \text{N}\cdot\text{m})(0.4\ \text{m})}{0.0795 \times 10^{-6}\ \text{m}^4} + \frac{(2250)(0.2)}{1.272 \times 10^{-6}} + \frac{(2250)(0.6)}{0.904 \times 10^{-6}}\right]$$

$$= 0.01634 + 0.00459 + 0.01939 = 0.0403\ \text{rad}$$

$$\phi_A = (0.0403\ \text{rad})\frac{360°}{2\pi\ \text{rad}} \qquad \phi_A = 2.31° \blacktriangleleft$$

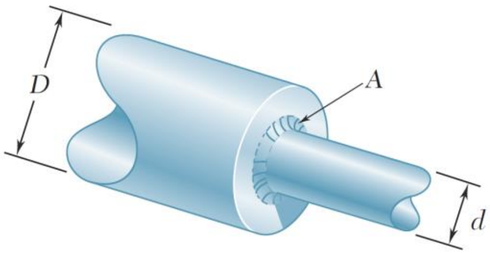


Fig. 3.28 Shaft with change in diameter.

be reduced through the use of a fillet, and the maximum value of the shearing stress at the fillet can be expressed as

$$\tau_{\max} = K\frac{Tc}{J} \tag{3.25}$$

where the stress Tc/J is the stress computed for the smaller-diameter shaft, and where K is a stress-concentration factor. Since the factor K depends only upon the ratio of the two diameters and the ratio of the radius of the fillet to the diameter of the smaller shaft, it may be computed once and for all and recorded in the form of a table or a graph, as shown in Fig. 3.29. We should note, however, that this procedure for determining localized shearing stresses is valid only as long as the value of $\tau_{\max}$ given by Eq. (3.25) does not exceed the proportional limit of the material, since the values of K plotted in Fig. 3.29 were obtained under the assumption of a linear relation between shearing stress and shearing strain. If plastic deformations occur, they will result in values of the maximum stress lower than those indicated by Eq. (3.25).

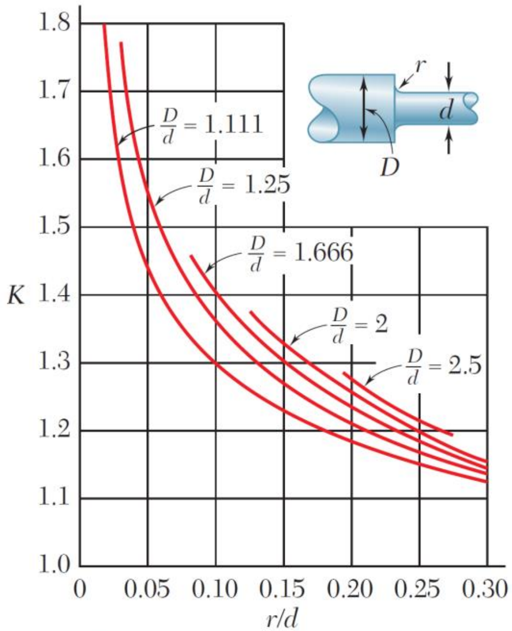


Fig. 3.29 Stress-concentration factors for fillets in circular shafts.†

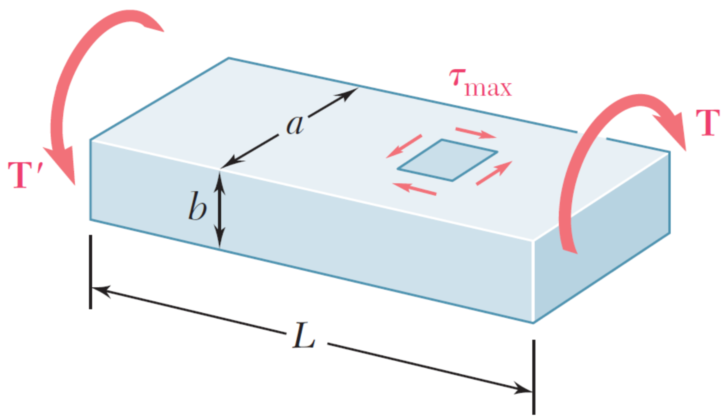


Fig. 3.45 Shaft with rectangular cross section.

TABLE 3.1. Coefficients for Rectangular Bars in Torsion

a/b	c_1	c_2
1.0	0.208	0.1406
1.2	0.219	0.1661
1.5	0.231	0.1958
2.0	0.246	0.229
2.5	0.258	0.249
3.0	0.267	0.263
4.0	0.282	0.281
5.0	0.291	0.291
10.0	0.312	0.312
∞	0.333	0.333

(Fig. 3.45), we find that the maximum shearing stress occurs along the center line of the *wider* face of the bar and is equal to

$$\tau_{max} = \frac{T}{c_1 a b^2} \tag{3.43}$$

The angle of twist, on the other hand, may be expressed as

$$\phi = \frac{TL}{c_2 a b^3 G} \tag{3.44}$$

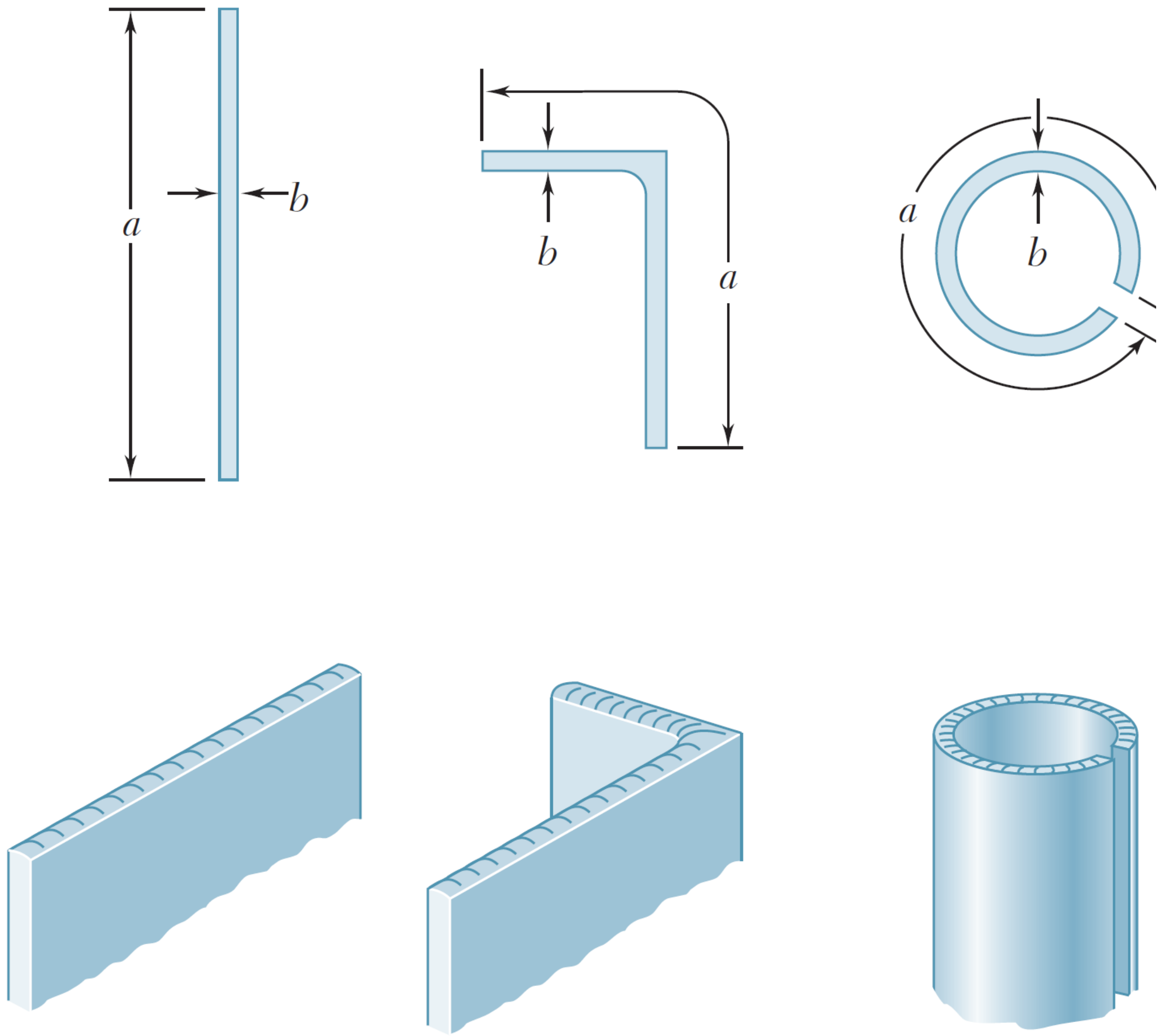


Fig. 3.47 Various thin-walled members.

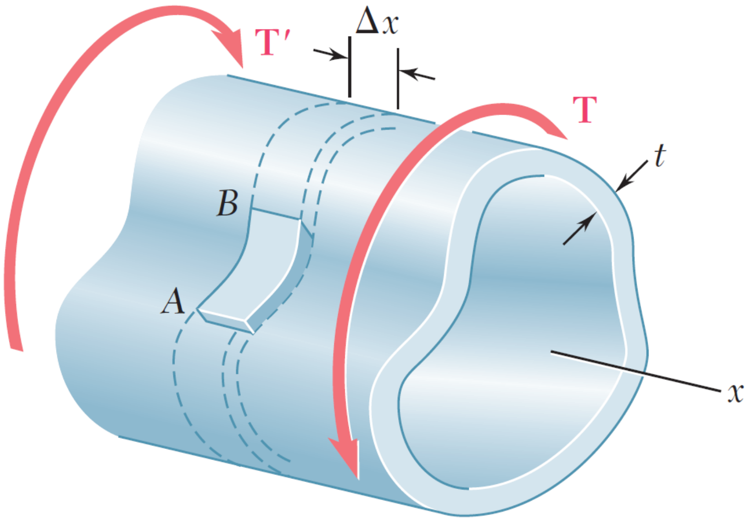


Fig. 3.48 Thin-walled hollow shaft.

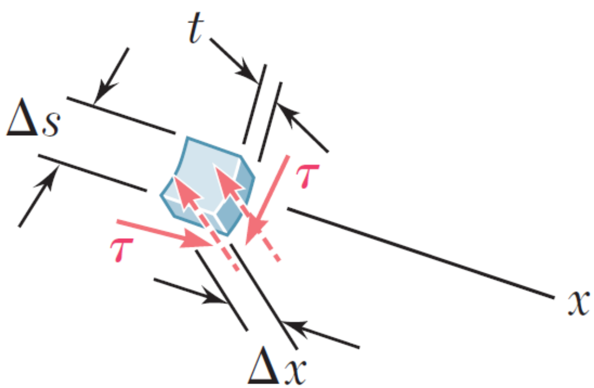


Fig. 3.50 Small element from segment.

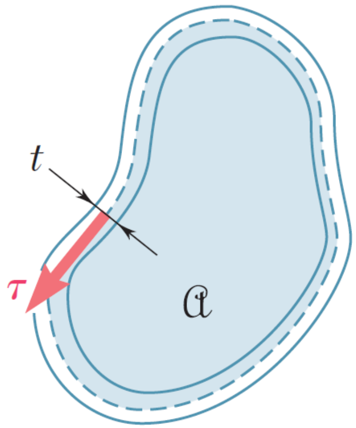


Fig. 3.54 Area for shear flow.

$$\tau = \frac{T}{2t\mathfrak{A}}$$

$$\phi = \frac{TL}{4\mathfrak{A}^2 G} \oint \frac{ds}{t}$$