VECTOR-VALUED CHARACTERS ON VECTOR-VALUED FUNCTION ALGEBRAS

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Abstract. Let A be a commutative unital Banach algebra and X be a compact space. We study the class of A-valued function algebras on X as subalgebras of $C(X,A)$ with certain properties. We introduce the notion of A-characters of an A-valued function algebra $\mathscr{A}$ as homomorphisms from $\mathscr{A}$ into A that basically have the same properties as evaluation homomorphisms $\mathcal{E}_x : f \mapsto f(x)$, with $x \in X$. We show that, under certain conditions, there is a one-to-one correspondence between the set of A-characters of $\mathscr{A}$ and the set of characters of the function algebra $\mathfrak{A} = \mathscr{A} \cap C(X)$ of all scalar-valued functions in $\mathscr{A}$. For the so-called natural A-valued function algebras, such as $C(X,A)$ and $\mathrm{Lip}(X,A)$, we show that $\mathcal{E}_x$ $(x \in X)$ are the only A-characters. Vector-valued characters are utilized to identify vector-valued spectra.

1. Introduction and Preliminaries

In this paper, we consider only commutative unital Banach algebras over the complex field $\mathbb{C}$ [3, 4, 11, 18].

Let A be a commutative Banach algebra. The set of all characters of A is denoted by $\mathfrak{M}(A)$. It is well-known that $\mathfrak{M}(A)$, equipped with the Gelfand topology, is a compact Hausdorff space called the character space of A. For every $a \in A$, let $\hat{a} : \mathfrak{M}(A) \to \mathbb{C}$, $\phi \mapsto \phi(a)$, be the Gelfand transform of a. The algebra A then can be seen, through its Gelfand representation $A \to C(\mathfrak{M}(A))$, $a \mapsto \hat{a}$, as a subalgebra of $C(\mathfrak{M}(A))$.

1.1. **Complex function algebras.** Let X be a compact Hausdorff space. The algebra $C(X)$ of all continuous complex-valued functions on X equipped with the uniform norm $\|\cdot\|_X$ is a commutative unital Banach algebra. A *function algebra* on X is a subalgebra $\mathfrak{A}$ of $C(X)$ that separates the points of X and contains the constant functions. A function algebra $\mathfrak{A}$ equipped with some complete algebra norm $\|\cdot\|$ is a *Banach function algebra*. If the norm of a Banach function algebra $\mathfrak{A}$ is equivalent to the uniform norm $\|\cdot\|_X$, then $\mathfrak{A}$ is a *uniform algebra*.

Let $\mathfrak{A}$ be a Banach function algebra on X. For every $x \in X$, the mapping $\varepsilon_x : \mathfrak{A} \to \mathbb{C}$, $f \mapsto f(x)$, is a character of $\mathfrak{A}$, and the mapping $J : X \to \mathfrak{M}(\mathfrak{A})$, $x \mapsto \varepsilon_x$, imbeds X homeomorphically as a compact subset of $\mathfrak{M}(\mathfrak{A})$. When J is surjective, one calls $\mathfrak{A}$ *natural* [4, Chapter 4]. For example, $C(X)$ is a natural uniform algebra, while, for the unit circle $\mathbb{T}$, the algebra $P(\mathbb{T})$ of all functions

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$f \in C(\mathbb{T})$ that can be approximated uniformly on $\mathbb{T}$ by polynomials is not natural Note, however, that every semisimple commutative unital Banach algebra A can be seen (through its Gelfand representation) as a natural Banach function algebra on $\mathfrak{M}(A)$. For more on function algebras see, for example, [6, 12].

1.2. **Vector-valued function algebras.** Let $(A, \|\cdot\|)$ be a commutative unital Banach algebra. The set of all A-valued continuous functions on X is denoted by $C(X, A)$. Algebraic operations on $C(X, A)$ are defined pointwise. The uniform norm $\|f\|_X$ of each function $f \in C(X, A)$ is defined in the obvious way. In this setting, $\big(C(X, A), \|\cdot\|_X\big)$ is a commutative unital Banach algebra.

Starting by Yood [17] in 1950, the character space of $C(X, A)$ has been studied by many authors. In 1957, Hausner [7] proved that τ is a character of $C(X, A)$ if, and only if, there exist a point $x \in X$ and a character $\phi \in \mathfrak{M}(A)$ such that $\tau(f) = \phi(f(x))$, for all $f \in C(X, A)$, whence $\mathfrak{M}(C(X, A))$ is homeomorphic to $X \times \mathfrak{M}(A)$. Recently, in [1], other characterizations of maximal ideals of $C(X, A)$ have been presented.

1.3. **Vector-valued characters.** Analogous with Banach function algebras, Banach A-valued function algebras are defined as subalgebras of $C(X, A)$ with certain properties (Definition 2.1). For a Banach A-valued function algebra $\mathscr{A}$ on X, consider the evaluation homomorphisms $\mathcal{E}_x : \mathscr{A} \to A$, $x \mapsto f(x)$, with $x \in X$. These A-valued homomorphisms are included in a certain class of homomorphisms that will be introduced and studied in Section 3 under the name of *vector-valued characters*. The set of all A-characters of $\mathscr{A}$ will be denoted by $\mathfrak{M}_A(\mathscr{A})$. Note that when $A = \mathbb{C}$, we have $C(X, A) = C(X)$. In this case, A-valued function algebras reduce to function algebras, and A-characters reduce to characters.

An application of vector-valued characters is presented in the forthcoming paper [2] to identify the vector-valued spectrum of functions $f \in \mathscr{A}$. It is known that the spectrum of an element $a \in A$ is equal to $\mathrm{SP}(a) = \{\phi(a) : \phi \in \mathfrak{M}(A)\}$. In [2] the A-valued spectrum $\vec{\mathrm{SP}}_A(f)$ of functions $f \in \mathscr{A}$ are studied and it is proved that, under certain conditions,

$$\vec{\mathrm{SP}}_A(f) = \{\Psi(f) : \Psi \in \mathfrak{M}_A(\mathscr{A})\}.$$

1.4. **Notations and conventions.** Since in this paper we are dealing with different types of functions and algebras, ambiguity may arise. Hence, a clear declaration of notations and conventions is given here.

(1) Throughout the paper, X is a compact Hausdorff space, and A is a commutative *semisimple* unital Banach algebra. The unit element of A is denoted by $\mathbf{1}$, and the set of invertible elements of A is denoted by $\mathrm{Inv}(A)$.
(2) If $f \in C(X)$ and $a \in A$, we write fa to denote the A-valued function $X \to A$, $x \mapsto f(x)a$. If $\mathfrak{A}$ is a function algebra on X, we let $\mathfrak{A}A$ be the linear span of $\{fa : f \in \mathfrak{A}, a \in A\}$. Hence, any element $f \in \mathfrak{A}A$ is of the form $f = f_1a_1 + \cdots + f_na_n$ with $f_j \in \mathfrak{A}$ and $a_j \in A$.
(3) Given an element $a \in A$, we use the same notation a for the constant function $X \to A$ given by $a(x) = a$, for all $x \in X$, and consider A as a closed subalgebra of $C(X, A)$. Since A is assumed to have a unit element $\mathbf{1}$, we identify $\mathbb{C}$ with

the closed subalgebra $\mathbb{C}\mathbf{1}$ of A, and thus every function $f : X \to \mathbb{C}$ can be seen as the A-valued function $X \to A$, $x \mapsto f(x)\mathbf{1}$; we use the same notation f for this A-valued function. In this regard, we admit the identification $C(X) = C(X)\mathbf{1}$ and consider $C(X)$ as a closed subalgebra of $C(X, A)$.

(4) To every continuous function $f : X \to A$, we correspond the function

$$\tilde{f} : \mathfrak{M}(A) \to C(X), \quad \tilde{f}(\phi) = \phi \circ f.$$

If $\mathcal{I}$ is a family of continuous A-valued functions on X, we define

$$\phi[\mathcal{I}] = \{\phi \circ f : f \in \mathcal{I}\} = \{\tilde{f}(\phi) : f \in \mathcal{I}\}.$$

2. Vector-valued Function Algebras

In this section, we introduce and study the notion of vector-valued function algebras.

Definition 2.1 (See [13])**.** Let X be a compact Hausdorff space, and $(A, \|\cdot\|)$ be a commutative unital Banach algebra. An *A-valued function algebra* on X is a subalgebra $\mathscr{A}$ of $C(X, A)$ such that (1) $\mathscr{A}$ contains all the constant functions $X \to A$, $x \mapsto a$, with $a \in A$, and (2) $\mathscr{A}$ separates the points of X in the sense that, for every pair x, y of distinct points in X, there exists $f \in A$ such that $f(x) \neq f(y)$. A *normed A-valued function algebra* on X is an A-valued function algebra $\mathscr{A}$ on X endowed with some algebra norm $|||\cdot|||$ such that the restriction of $|||\cdot|||$ to A is equivalent to the original norm $\|\cdot\|$ of A, and $\|f\|_X \leq |||f|||$, for every $f \in \mathscr{A}$. A complete normed A-valued function algebra is called a *Banach A-valued function algebra*. A Banach A-valued function algebra $\mathscr{A}$ is called an *A-valued uniform algebra* if the given norm of $\mathscr{A}$ is equivalent to the uniform norm $\|\cdot\|_X$.

If there is no risk of confusion, instead of $|||\cdot|||$, we use the same notation $\|\cdot\|$ for the norm of $\mathscr{A}$.

Let $\mathscr{A}$ be an A-valued function algebra on X. For every $x \in X$, define $\mathcal{E}_x : \mathscr{A} \to A$ by $\mathcal{E}_x(f) = f(x)$. We call $\mathcal{E}_x$ the *evaluation homomorphism* at x. Our definition of Banach A-valued function algebras implies that every evaluation homomorphism $\mathcal{E}_x$ is continuous. As it is mentioned in [13], if the condition $\|f\|_X \leq |||f|||$, for all $f \in \mathscr{A}$, is replaced by the requirement that every evaluation homomorphism $\mathcal{E}_x$ is continuous, then one can find some constant M such that

$$\|f\|_X \leq M|||f||| \quad (f \in \mathscr{A}).$$

2.1. Admissible function algebras.

Given a complex-valued function algebra $\mathfrak{A}$ and an A-valued function algebra $\mathscr{A}$ on X, according to [13, Definition 2.1], the quadruple $(X, A, \mathfrak{A}, \mathscr{A})$ is *admissible* if $\mathfrak{A}$ is natural, $\mathfrak{A}A \subset \mathscr{A}$, and

$$\{\phi \circ f : \phi \in \mathfrak{M}(A), f \in \mathscr{A}\} \subset \mathfrak{A}.$$

Taking this into account, we make the following definition.

Definition 2.2. An A-valued function algebra $\mathscr{A}$ is said to be *admissible* if

$$\{(\phi \circ f)\mathbf{1} : \phi \in \mathfrak{M}(A), f \in \mathscr{A}\} \subset \mathscr{A}. \tag{2.1}$$

When $\mathscr{A}$ is admissible, we set $\mathfrak{A} = \mathscr{A} \cap C(X)$ (more precisely, $\mathscr{A} \cap C(X)\mathbf{1}$). Then $\mathfrak{A}$ is the subalgebra of $\mathscr{A}$ consisting of all scalar-valued functions in $\mathscr{A}$ and forms a function algebra by itself. Note that $\mathfrak{A} = \phi[\mathscr{A}]$, for all $\phi \in \mathfrak{M}(A)$. Of course, if $(X, A, \mathfrak{A}, \mathscr{A})$ is an admissible quadruple, in the sense of [13], then $\mathscr{A}$ satisfies (2.1) and $\mathfrak{A} = \mathscr{A} \cap C(X)$ is natural. In general, however, we do not assume $\mathfrak{A}$ to be natural, hence an admissible A-valued function algebra $\mathscr{A}$ may not form an admissible quadruple.

Example 2.3. Let $\mathfrak{A}$ be a complex-valued function algebra on X. Then $\mathfrak{A}A$ is an admissible A-valued function algebra on X. Hence, the uniform closure of $\mathfrak{A}A$ in $C(X, A)$ is an admissible A-valued uniform algebra (Proposition 2.5 below).

Other examples of admissible function algebras are presented in Section 4. Here, we present an example to show that not all vector-valued function algebras are admissible.

Example 2.4. Let K be a compact subset of $\mathbb{C}$ which is not polynomially convex so that $P(K) \neq R(K)$. For example, let $K = \mathbb{T}$ be the unit circle. Set

$$\mathscr{A} = \{(f_p, f_r) : f_p \in P(K), f_r \in R(K)\}.$$

Then $\mathscr{A}$ is a uniformly closed subalgebra of $C(K, \mathbb{C}^2)$, it contains all the constant functions $(\alpha, \beta) \in \mathbb{C}^2$ and separates the points of K. Hence $\mathscr{A}$ is a $\mathbb{C}^2$-valued uniform algebra on K. Let $\mathbf{1} = (1, 1)$ be the unit element of $\mathbb{C}^2$, and let π_1 and π_2 be the coordinate projections of $\mathbb{C}^2$. Then $\mathfrak{M}(\mathbb{C}^2) = \{\pi_1, \pi_2\}$, and

$$\pi_1[\mathscr{A}]\mathbf{1} = \{f\mathbf{1} : f \in P(K)\} = \{(f, f) : f \in P(K)\},$$
$$\pi_2[\mathscr{A}]\mathbf{1} = \{f\mathbf{1} : f \in R(K)\} = \{(f, f) : f \in R(K)\}.$$

We see that $\pi_1[\mathscr{A}]\mathbf{1} \subset \mathscr{A}$ while $\pi_2[\mathscr{A}]\mathbf{1} \not\subset \mathscr{A}$. Hence $\mathscr{A}$ is not admissible.

Proposition 2.5. *Let $\mathscr{A}$ be an admissible A-valued function algebra on X, with $\mathfrak{A} = \mathscr{A} \cap C(X)$. Then the uniform closure $\bar{\mathscr{A}}$ is an admissible A-valued uniform algebra on X with $\bar{\mathfrak{A}} = C(X) \cap \bar{\mathscr{A}}$.*

Proof. The fact that $\bar{\mathscr{A}}$ is an A-valued uniform algebra is clear. The inclusion $\bar{\mathfrak{A}} \subset C(X) \cap \bar{\mathscr{A}}$ is also obvious. Take $f \in C(X) \cap \bar{\mathscr{A}}$. Then, there exists a sequence $\{f_n\}$ of A-valued functions in $\mathscr{A}$ such that $f_n \to f$ uniformly on X. For some $\phi \in \mathfrak{M}(A)$, take $g_n = \phi \circ f_n$. Then $g_n \in \mathfrak{A}$ and, since $f = (\phi \circ f)\mathbf{1}$, we have

$$\|g_n - f\|_X = \|\phi \circ f_n - \phi \circ f\|_X \leq \|f_n - f\|_X \to 0.$$

Therefore, $g_n \to f$ uniformly on X and thus $f \in \bar{\mathfrak{A}}$. $\square$

2.2. **Certain vector-valued uniform algebras.** Let $\mathfrak{A}_0$ be a complex function algebra on X, and let $\mathfrak{A}$ and $\mathscr{A}$ be the uniform closures of $\mathfrak{A}_0$ and $\mathfrak{A}_0 A$, in $C(X)$ and $C(X, A)$, respectively. Then $\mathscr{A}$ is an admissible A-valued uniform algebra on X with $\mathfrak{A} = C(X) \cap \mathscr{A}$. The algebra $\mathscr{A}$ is isometrically isomorphic to the injective tensor product $\mathfrak{A} \hat{\otimes}_\epsilon A$ (cf. [11, Proposition 1.5.6]). To see this, let $T : \mathfrak{A} \otimes A \to \mathscr{A}$ be the unique linear mapping, given by [3, Theorem 42.6], such that $T(f \otimes a)(x) = f(x)a$, for all $x \in X$. Let $\mathfrak{A}_1^*$ and A_1^* denote the closed unit ball of $\mathfrak{A}^*$ and A^*, respectively, and let $\|\cdot\|_\epsilon$ denote the injective tensor norm. Then

$$\begin{aligned}\Big\|T\Big(\sum_{i=1}^{n} f_i\otimes a_i\Big)\Big\|_X &= \sup_{x\in X}\Big\|\sum_{i=1}^{n} f_i(x)a_i\Big\| = \sup_{x\in X}\sup_{\nu\in A_1^*}\Big|\sum_{i=1}^{n} f_i(x)\nu(a_i)\Big|\\ &= \sup_{\nu\in A_1^*}\Big\|\sum_{i=1}^{n} f_i(\cdot)\nu(a_i)\Big\|_X = \sup_{\nu\in A_1^*}\sup_{\mu\in\mathfrak{A}_1^*}\Big|\mu\Big(\sum_{i=1}^{n} f_i(\cdot)\nu(a_i)\Big)\Big|\\ &= \sup_{\mu\in\mathfrak{A}_1^*}\sup_{\nu\in A_1^*}\Big|\sum_{i=1}^{n}\mu(f_i)\nu(a_i)\Big| = \Big\|\sum_{i=1}^{n} f_i\otimes a_i\Big\|_\epsilon.\end{aligned}$$

Hence T extends to an isometry $\overline{T}$ from $\mathfrak{A}\,\hat{\otimes}_\epsilon\, A$ into $\mathscr{A}$. Since the range of T contains $\mathfrak{A}_0A$, which is dense in $\mathscr{A}$, the range of $\overline{T}$ is the whole of $\mathscr{A}$. We remark that, by a theorem of Tomiyama [16], the character space $\mathfrak{M}(\mathscr{A})$ of $\mathscr{A}$ is homeomorphic to $\mathfrak{M}(\mathfrak{A})\times\mathfrak{M}(A)$.

Let K be a compact subset of $\mathbb{C}$. Associated with K there are three vector-valued uniform algebras in which we are interested. Let $P_0(K,A)$ be the algebra of the restriction to K of A-valued polynomials $p(z)=a_nz^n+\cdots+a_1z+a_0$ with coefficients in A. Let $R_0(K,A)$ be the algebra of the restriction to K of rational functions of the form $p(z)/q(z)$, where $p(z)$ and $q(z)$ are A-valued polynomials, and $q(\lambda)\in \mathrm{Inv}(A)$ for $\lambda\in K$. And let $H_0(K,A)$ be the algebra of A-valued functions on K having a holomorphic extension to a neighbourhood of K.

When $A=\mathbb{C}$, we drop A and write $P_0(K)$, $R_0(K)$ and $H_0(K)$. Their uniform closures in $C(K)$, denoted by $P(K)$, $R(K)$ and $H(K)$, are complex uniform algebras (for more on complex uniform algebras, see [6] or [12]).

The algebras $P_0(K,A)$, $R_0(K,A)$ and $H_0(K,A)$ are admissible A-valued function algebras on K, and their uniform closures in $C(K,A)$, denoted by $P(K,A)$, $R(K,A)$ and $H(K,A)$, are admissible A-valued uniform algebras. It obvious that

$$P(K,A)\subset R(K,A)\subset H(K,A).$$

Every polynomial $p(z)=a_0+a_1z+\cdots+a_nz^n$ in $P_0(K,A)$ is, clearly, of the form $p(z)=p_0(z)a_0+p_1(z)a_1+\cdots+p_n(z)a^n$, where $p_0,p_1,\ldots,p_n$ are polynomials in $P_0(K)$. Thus $P_0(K,A)=P_0(K)A$. The above discussion shows that $P(K,A)$ is isometrically isomorphic to $P(K)\,\hat{\otimes}_\varepsilon\, A$. The character space of $P(K)$ is homeomorphic to $\hat{K}$, the polynomially convex hull of K ([12, Section 5.2]). The character space of $P(K,A)$ is, therefore, homeomorphic to $\hat{K}\times\mathfrak{M}(A)$.

Runge's classical approximation theorem states that if Λ is a subset of $\mathbb{C}$ such that Λ has nonempty intersection with each bounded component of $\mathbb{C}\setminus K$, then every function $f\in H_0(K)$ can be approximated uniformly on K by rational functions with poles only among the points of Λ and at infinity ([3, Theorem 7.7]). In particular, $R(K)=H(K)$. The following is a version of Runge's theorem for vector-valued functions.

Theorem 2.6 (Runge)**.** *Let K be a compact subset of $\mathbb{C}$, and let Λ be a subset of $\mathbb{C}\setminus K$ having nonempty intersection with each bounded component of $\mathbb{C}\setminus K$. Then every function $f\in H_0(K,A)$ can be approximated uniformly on K by A-valued* *rational functions of the form*

$$r(z) = r_1(z)a_1 + r_2(z)a_2 + \cdots + r_n(z)a_n, \tag{2.2}$$

where $r_i(z)$, for $1 \le i \le n$, are rational functions in $R_0(K)$ with poles only among the points of Λ and at infinity, and $a_1, a_2, \ldots, a_n \in A$.

Proof. Take $f \in H_0(K, A)$. Then, there exists an open set D such that $K \subset D$ and $f : D \to A$ is holomorphic. We use the same notation as in [3]. We let E be a punched disc envelope for (K, D); see [3, Definition 6.2]. The Cauchy theorem and the Cauchy integral formula are also valid for Banach space valued holomorphic functions (see the remark after Corollary 6.6 in [3] and [14, Theorem 3.31]). Therefore,

$$f(z) = \frac{1}{2\pi \mathrm{i}} \int_{\partial E} \frac{f(s)}{s - z} ds \quad (z \in K).$$

Then, by [3, Proposition 6.5], one can write

$$f(z) = \sum_{n=0}^{\infty} \alpha_n (z - z_0)^n + \sum_{j=1}^{m} \sum_{n=1}^{\infty} \frac{\beta_{jn}}{(z - z_j)^n} \quad (z \in K), \tag{2.3}$$

where $z_0 \in \mathbb{C}$, $z_1, z_2, \ldots, z_m \in \mathbb{C} \setminus K$, and the coefficients α_n, β_{jn} belong to A. Note that the series in (2.3) converges uniformly on K.

So far, we have seen that f can be approximated uniformly on K by A-valued rational functions of the form (2.2), where $r_i(z)$, for $1 \le i \le n$, are rational functions in $R_0(K)$ with poles just outside K. Using Runge's classical theorem, each $r_i(z)$ can be approximated uniformly on K by rational functions with poles only among the points of Λ and at infinity. Hence, we conclude that f can be approximated uniformly on K by rational functions of the form (2.2) with preassigned poles. □

As a consequence of the above theorem, we see that the uniform closures of $R_0(K)A$, $H_0(K)A$, $R_0(K, A)$ and $H_0(K, A)$ are all the same. In particular,

$$R(K, A) = H(K, A).$$

Corollary 2.7. *The algebra $R(K, A)$ is isometrically isomorphic to $R(K) \hat{\otimes}_\varepsilon A$ and, therefore, $\mathfrak{M}(R(K, A))$ is homeomorphic to $K \times \mathfrak{M}(A)$.*

We remark that the equality $\mathfrak{M}(R(K, A)) = K \times \mathfrak{M}(A)$ is proved in [13]. The authors, however, did not notice the equality $R(K, A) = R(K) \hat{\otimes}_\varepsilon A$.

3. Vector-valued Characters

Let $\mathscr{A}$ be a Banach A-valued function algebra on X, and consider the point evaluation homomorphisms $\mathcal{E}_x : \mathscr{A} \to A$. These kind of homomorphisms enjoy the following properties:

- $\mathcal{E}_x(a) = a$, for all $a \in A$,
- $\mathcal{E}_x(\phi \circ f) = \phi(\mathcal{E}_x f)$, for all $f \in \mathscr{A}$ and $\phi \in \mathfrak{M}(A)$,
- If $\mathscr{A}$ is admissible (with $\mathfrak{A} = C(X) \cap \mathscr{A}$) then $\mathcal{E}_x|_{\mathfrak{A}}$ is a character of $\mathfrak{A}$, namely, the evaluation character ε_x.

We now introduce the class of all homomorphisms from $\mathscr{A}$ into A having the same properties as the point evaluation homomorphisms $\mathcal{E}_x$ $(x \in X)$.

Definition 3.1. Let $\mathscr{A}$ be an admissible A-valued function algebra on X. A homomorphism $\Psi : \mathscr{A} \to A$ is called an *A-character* if $\Psi(\mathbf{1}) = \mathbf{1}$ and $\phi(\Psi f) = \Psi(\phi \circ f)$, for all $f \in \mathscr{A}$ and $\phi \in \mathfrak{M}(A)$. The set of all A-characters of $\mathscr{A}$ is denoted by $\mathfrak{M}_A(\mathscr{A})$.

That every A-character $\Psi : \mathscr{A} \to A$ satisfies $\Psi(a) = a$, for all $a \in A$, is easy to see. In fact, since $\phi(\Psi(a)) = \Psi(\phi(a)) = \phi(a)$, for all $\phi \in \mathfrak{M}(A)$, and A is semisimple, we get $\Psi(a) = a$.

Proposition 3.2. *Let $\Psi : \mathscr{A} \to A$ be a linear operator such that $\Psi(\mathbf{1}) = \mathbf{1}$ and $\phi(\Psi f) = \Psi(\phi \circ f)$, for all $f \in \mathscr{A}$ and $\phi \in \mathfrak{M}(A)$. Then, the following are equivalent:*

(i) *Ψ is an A-character,*
(ii) *$\Psi(f) \neq \mathbf{0}$, for every $f \in \mathrm{Inv}(\mathscr{A})$,*
(iii) *$\Psi(f) \neq \mathbf{0}$, for every $f \in \mathrm{Inv}(\mathfrak{A})$,*
(iv) *if $\psi = \Psi|_{\mathfrak{A}}$, then $\psi \in \mathfrak{M}(\mathfrak{A})$.*

Proof. (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) is clear. The implication (iii) $\Rightarrow$ (iv) follows from [14, Theorem 10.9]. To prove (iv) $\Rightarrow$ (i), take $f, g \in \mathscr{A}$. For every $\phi \in \mathfrak{M}(A)$, we have

$$\phi(\Psi(fg)) = \Psi(\phi \circ fg) = \psi((\phi \circ f)(\phi \circ g)) = \psi(\phi \circ f)\psi(\phi \circ g) = \phi(\Psi(f)\Psi(g)).$$

Since A is semisimple, we get $\Psi(fg) = \Psi(f)\Psi(g)$. $\square$

Every A-character is automatically continuous by Johnson's theorem [10]. If A is a uniform algebra and $\mathscr{A}$ is an A-valued uniform algebra, we even have $\|\Psi\| = 1$, for any A-character Ψ.

Proposition 3.3. *Let Ψ_1 and Ψ_2 be A-characters on $\mathscr{A}$, and set $\psi_1 = \Psi_1|_{\mathfrak{A}}$ and $\psi_2 = \Psi_2|_{\mathfrak{A}}$. The following are equivalent:*

(i) $\Psi_1 = \Psi_2$, (ii) $\ker \Psi_1 = \ker \Psi_2$, (iii) $\ker \psi_1 = \ker \psi_2$, (iv) $\psi_1 = \psi_2$.

Proof. The implication (i) $\Rightarrow$ (ii) is obvious. The implication (ii) $\Rightarrow$ (iii), follows from the fact that $\ker \psi_i = \ker \Psi_i \cap \mathfrak{A}$, for $i = 1, 2$. The implication (iii) $\Rightarrow$ (iv) follows form [14, Theorem 11.5]. Finally, if we have (iv), then, for every $f \in \mathscr{A}$,

$$\phi(\Psi_1(f)) = \psi_1(\phi(f)) = \psi_2(\phi(f)) = \phi(\Psi_2(f)) \quad (\phi \in \mathfrak{M}(A)).$$

Since A is semisimple, we get $\Psi_1(f) = \Psi_2(f)$, for all $f \in \mathscr{A}$. $\square$

Definition 3.4. Let $\mathscr{A}$ be an admissible A-valued function algebra on X. Given a character $\psi \in \mathfrak{M}(\mathfrak{A})$, if there exists an A-character Ψ on $\mathscr{A}$ such that $\Psi|_{\mathfrak{A}} = \psi$, then we say that ψ *lifts* to the A-character Ψ.

Proposition 3.3 shows that, if $\psi \in \mathfrak{M}(\mathfrak{A})$ lifts to Ψ_1 and Ψ_2, then $\Psi_1 = \Psi_2$. For every $x \in X$, the unique A-character to which the evaluation character ε_x lifts is the evaluation homomorphism $\mathcal{E}_x$. In the following, we investigate conditions under which every character $\psi \in \mathfrak{M}(\mathfrak{A})$ lifts to some A-character Ψ. To proceed, we need some definitions, notations and auxiliary results.

Let $\mathscr{A}$ be an admissible Banach A-valued function algebra on X and $\mathfrak{A} = \mathscr{A} \cap C(X)\mathbf{1}$. Then $\mathfrak{A}$ is a Banach function algebra. For every $f \in \mathscr{A}$, consider the function $\tilde{f} : \mathfrak{M}(A) \to \mathfrak{A}$, $\phi \mapsto \phi \circ f$. Set $\mathcal{X} = \mathfrak{M}(A)$ and $\tilde{\mathscr{A}} = \{\tilde{f} : f \in \mathscr{A}\}$. Suppose every $\tilde{f}$ is continuous, with respect to the Gelfand topology of $\mathfrak{M}(A)$ and the norm topology of $\mathfrak{A}$ (this is the case for uniform algebras; see Corollary 3.6). Then $\tilde{\mathscr{A}}$ is an $\mathfrak{A}$-valued function algebra on $\mathcal{X}$. In Theorem 3.8, we will discuss conditions under which $\tilde{\mathscr{A}}$ is admissible; we will see that $\tilde{\mathscr{A}}$ is admissible if and only if every character $\psi \in \mathfrak{M}(\mathfrak{A})$ lifts to some A-character $\Psi \in \mathfrak{M}_A(\mathscr{A})$.

In the following, we extend $\tilde{f}$ to a mapping from A^* to $C(X)$, and we still denote this extension is by $\tilde{f}$. Note that $\phi \circ f \in C(X)$, for all $\phi \in A^*$, and $\|\phi \circ f\|_X \le \|\phi\| \|f\|_X$.

Proposition 3.5. *With respect to the w*-topology of A^* and the uniform topology of $C(X)$, every mapping $\tilde{f} : A^* \to C(X)$ is continuous on bounded subsets of A^*.*

Proof. Let $\{\phi_\alpha\}$ be a net in A^* that converges, in the w*-topology, to some $\phi_0 \in A^*$ and suppose $\|\phi_\alpha\| \le M$, for all α. Take $\varepsilon > 0$ and set, for every $x \in X$,

$$V_x = \{s \in X : \|f(s) - f(x)\| < \varepsilon\}.$$

Then $\{V_x : x \in X\}$ is an open covering of the compact space X. Hence, there exist finitely many points $x_1, \ldots, x_n$ in X such that $X \subset V_{x_1} \cup \cdots \cup V_{x_n}$. Set

$$U_0 = \bigl\{\phi \in A^* : \bigl|\phi\bigl(f(x_i)\bigr) - \phi_0\bigl(f(x_i)\bigr)\bigr| < \varepsilon,\ 1 \le i \le n\bigr\}.$$

The set U_0 is an open neighbourhood of ϕ_0 in the w*-topology. Since $\phi_\alpha \to \phi_0$, there exists α_0 such that $\phi_\alpha \in U_0$ for $\alpha \ge \alpha_0$. If $x \in X$, then $\|f(x) - f(x_i)\| < \varepsilon$, for some $i \in \{1, \ldots, n\}$, and thus, for $\alpha \ge \alpha_0$,

$$\begin{aligned} |\phi_\alpha \circ f(x) - \phi_0 \circ f(x)| &\le |\phi_\alpha(f(x)) - \phi_\alpha(f(x_i))| + |\phi_\alpha(f(x_i)) - \phi_0(f(x_i))| \\ &\quad + |\phi_0(f(x_i)) - \phi_0(f(x))| \\ &< M\varepsilon + \varepsilon + \|\phi_0\|\varepsilon. \end{aligned}$$

Since $x \in X$ is arbitrary, we get $\|\phi_\alpha \circ f - \phi_0 \circ f\|_X \le \varepsilon(M + \|\phi_0\| + 1)$. □

Since $\mathfrak{M}(A)$ is a bounded subset of A^*, we get the following result for uniform algebras.

Corollary 3.6. *Let $\mathscr{A}$ be an admissible A-valued uniform algebra on A. Then $\tilde{f} \in C(\mathfrak{M}(A), \mathfrak{A})$, for every $f \in \mathscr{A}$, and $\tilde{\mathscr{A}} = \{\tilde{f} : f \in \mathscr{A}\}$ is an $\mathfrak{A}$-valued uniform algebra on $\mathfrak{M}(A)$.*

To prove our main result, we also need the following lemma.

Lemma 3.7. *For every $f \in \mathscr{A}$, if $\tilde{f} : \mathfrak{M}(A) \to \mathfrak{A}$ is scalar-valued, then f is a constant function and, therefore, $\tilde{f} = \hat{a}$, for some $a \in A$.*

Proof. Fix a point $x_0 \in A$ and let $a = f(x_0)$. The function $\tilde{f}$ being scalar-valued means that, for every $\phi \in \mathfrak{M}(A)$, there is a complex number λ such that $\tilde{f}(\phi) = \phi \circ f = \lambda$. This means that $\phi \circ f$ is a constant function on X so that

$$\phi(f(x)) = \phi(f(x_0)) = \phi(a) \quad (x \in X). \tag{3.1}$$

Since A is semisimple and (3.1) holds for every $\phi \in \mathfrak{M}(A)$, we must have $f(x) = a$, for all $x \in X$. Thus, $\tilde{f} = \hat{a}$. □

We are now ready to state and prove the main result of the section.

Theorem 3.8. *Let $\mathscr{A}$ be an admissible Banach A-valued function algebra on X, and let E be the linear span of $\mathfrak{M}(A)$ in A^*. The following statements are equivalent.*

(i) *for every $\psi \in \mathfrak{M}(\mathfrak{A})$ and $f \in \mathscr{A}$, the mapping $g : E \to \mathbb{C}$, defined by $g(\phi) = \psi(\phi \circ f)$, is continuous with respect to the w^*-topology of E;*

(ii) *every $\psi \in \mathfrak{M}(\mathfrak{A})$ lifts to an A-character $\Psi : \mathscr{A} \to A$;*

(iii) *every $f \in \mathscr{A}$ has a unique extension $F : \mathfrak{M}(\mathfrak{A}) \to A$ such that*

$$\phi(F(\psi)) = \psi(\phi \circ f) \quad (\psi \in \mathfrak{M}(\mathfrak{A}),\ \phi \in \mathfrak{M}(A));$$

Moreover, if the functions $\tilde{f} : \mathfrak{M}(A) \to \mathfrak{A}$, where $f \in \mathscr{A}$, are all continuous so that $\tilde{\mathscr{A}} = \{\tilde{f} : f \in \mathscr{A}\}$ is an $\mathfrak{A}$-valued function algebra on $\mathfrak{M}(A)$, then the above statements are equivalent to

(iv) *$\tilde{\mathscr{A}}$ is admissible.*

Proof. (i) $\Rightarrow$ (ii): Fix $\psi \in \mathfrak{M}(\mathfrak{A})$ and $f \in \mathscr{A}$. Since $\phi \circ f \in \mathfrak{A}$, for every $\phi \in E$, we see that g is a well-defined linear functional on E. Endowed with the w*-topology, A^* is a locally convex space with A as its dual. Since g is w*-continuous, by the Hahn-Banach extension theorem [14, Theorem 3.6], there is a w*-continuous linear functional G on A^* that extends g. Hence $G = \hat{a}$, for some $a \in A$, and since A is semisimple, a is unique. Now, define $\Psi(f) = a$. Then

$$\phi(\Psi(f)) = \hat{a}(\phi) = g(\phi) = \psi(\phi \circ f) \quad (\phi \in \mathfrak{M}(A)).$$

It is easily seen that $\Psi : \mathscr{A} \to A$ is an A-character and $\Psi|_{\mathfrak{A}} = \psi$.

(ii) $\Rightarrow$ (iii): Fix $f \in \mathscr{A}$ and define $F : \mathfrak{M}(\mathfrak{A}) \to A$ by $F(\psi) = \Psi(f)$, where Ψ is the unique A-character of $\mathscr{A}$ to which ψ lifts. Considering the identification $x \mapsto \varepsilon_x$ and the fact that each ε_x lifts to $\mathcal{E}_x : f \mapsto f(x)$, we get

$$F(x) = F(\varepsilon_x) = \mathcal{E}_x(f) = f(x) \quad (x \in X).$$

So, $F|_X = f$. Also, $\phi(F(\psi)) = \phi(\Psi(f)) = \psi(\phi \circ f)$.

(iii) $\Rightarrow$ (i): Fix $\psi \in \mathfrak{M}(\mathfrak{A})$ and $f \in \mathscr{A}$, and put $a = F(\psi)$, where F is the unique extension of f to $\mathfrak{M}(\mathfrak{A})$ given by (iii). Then $g(\phi) = \hat{a}(\phi)$, for every $\phi \in E$, which is, obviously, a continuous function with respect to the w*-topology of E.

Finally, suppose $\tilde{\mathscr{A}} = \{\tilde{f} : f \in \mathscr{A}\}$ is an $\mathfrak{A}$-valued function algebra on $\mathfrak{M}(A)$. We prove (ii) $\Rightarrow$ (iv) $\Rightarrow$ (i). If $\mathscr{A}$ satisfies (ii), then

$$\psi \circ \tilde{f} = \widehat{\Psi(f)} \in \hat{A} \subset \tilde{\mathscr{A}}, \quad (f \in \mathscr{A},\ \psi \in \mathfrak{M}(\mathfrak{A})).$$

This means that $\{\psi \circ \tilde{f} : \psi \in \mathfrak{M}(\mathfrak{A}),\ f \in \mathscr{A}\} \subset \tilde{\mathscr{A}}$ and thus $\tilde{\mathscr{A}}$ is admissible. Conversely, suppose $\tilde{\mathscr{A}}$ is admissible. Then, for every $\psi \in \mathfrak{M}(\mathfrak{A})$ and $f \in \mathscr{A}$, the function $\psi \circ \tilde{f}$ is a complex-valued function in $\tilde{\mathscr{A}}$. By Lemma 3.7, $\psi \circ \tilde{f}$ belongs to $\hat{A}$ so that $\psi \circ \tilde{f} = \hat{a}$, for some $a \in A$. Hence, $g(\phi) = \hat{a}(\phi)$, for every $\phi \in E$, and g is w*-continuous. □

The following shows that for a wide class of admissible A-valued function algebras, including admissible A-valued uniform algebras, the mapping g in the above theorem is continuous and thus every $\psi \in \mathfrak{M}(\mathfrak{A})$ lifts to some $\Psi \in \mathfrak{M}_A(\mathscr{A})$.

Theorem 3.9. *Let $\mathscr{A}$ be an admissible Banach A-valued function algebra on X with $\mathfrak{A} = C(X) \cap \mathscr{A}$. If $\|\hat{f}\| = \|f\|_X$, for all $f \in \mathfrak{A}$, then the mapping g in Theorem 3.8 is continuous and thus every character $\psi \in \mathfrak{M}(\mathfrak{A})$ lifts to some A-character $\Psi \in \mathfrak{M}_A(\mathscr{A})$. In particular, if $\mathfrak{A}$ is a uniform algebra, then $\mathscr{A}$ satisfies all conditions in Theorem 3.8.*

Proof. Take $\psi \in \mathfrak{M}(\mathfrak{A})$ and let g be as in Theorem 3.8. Since $\|\hat{f}\| = \|f\|_X$, for every $f \in \mathfrak{A}$, ψ is a continuous functional on $(\mathfrak{A}, \|\cdot\|_X)$; see [8]. By the Hahn-Banach theorem, ψ extends to a continuous linear functional $\bar{\psi}$ on $C(X)$. This, in turn, implies that g extends to a linear functional $\bar{g} : A^* \to \mathbb{C}$ defined by $\bar{g}(\phi) = \bar{\psi}(\phi \circ f)$. By Proposition 3.5, the extended mapping $\tilde{f} : A^* \to C(X)$ is w*-continuous on bounded subsets of A^*. Hence, the linear functional $\bar{g}$ is w*-continuous on bounded subsets of A^*. Since A is a Banach space, Corollary 3.11.4 in [9] shows that $\bar{g}$ is w*-continuous on A^*. ☐

Corollary 3.10. *If $\mathscr{A}$ is an admissible A-valued uniform algebra on X, then $\tilde{\mathscr{A}}$ is an admissible $\mathfrak{A}$-valued uniform algebra on $\mathfrak{M}(A)$.*

When $\mathscr{A}$ is an admissible A-valued uniform algebra, every $f \in \mathscr{A}$ extends to a function $F : \mathfrak{M}(\mathfrak{A}) \to A$. If, in addition, A is a uniform algebra, one can prove that this extension F is continuous and the following maximum principle holds;

$$\|F\|_{\mathfrak{M}(\mathfrak{A})} = \|F\|_X = \|f\|_X.$$

Remark. When $\mathfrak{A}$ is uniformly closed in $C(X)$, every linear functional $\psi \in \mathfrak{A}^*$ lifts to some bounded linear operator $\Psi : \mathscr{A} \to A$ with the property that $\phi(\Psi f) = \psi(\phi \circ f)$, for all $\phi \in A^*$. In fact, by the Hahn-Banach theorem, ψ extends to a linear functional $\bar{\psi} \in C(X)^*$. By the Riesz representation theorem, there is a complex Radon measure μ on X such that $\psi(f) = \int_X f d\mu$, for all $f \in \mathfrak{A}$. Using [14, Theorem 3.7], one can define $\Psi(f) = \int_X f d\mu$, for every $f \in \mathscr{A}$, such that

$$\phi(\Psi(f)) = \int_X (\phi \circ f) d\mu = \psi(\phi \circ f) \quad (f \in \mathscr{A},\, \phi \in A^*).$$

4. Examples

We conclude the paper by giving some examples of identifying the A-characters of certain admissible Banach A-valued function algebras.

Example 4.1. Let $\mathscr{A} = C(X, A)$. Then $\mathfrak{A} = C(X)$ is natural, that is, its only characters are the point evaluation characters ε_x $(x \in X)$. Hence the only A-characters of $C(X, A)$ are the point evaluation homomorphisms $\mathcal{E}_x$ $(x \in X)$.

Another example is $\mathscr{A} = R(K, A)$, where $K \subset \mathbb{C}$ is compact. In this case $\mathfrak{A} = R(K)$ is also natural. Hence the only A-characters of $R(K, A)$ are the point evaluation homomorphisms $\mathcal{E}_\lambda$ $(\lambda \in K)$.

Example 4.2. Let K be a compact subset of $\mathbb{C}$ and let $\mathscr{A} = P(K, A)$. Then $\mathfrak{M}(P(K)) = \hat{K}$, the polynomially convex hull of K. Since $\mathscr{A}$ is an A-valued uniform algebra, by Theorem 3.9, every $f \in P(K, A)$ extends to a function $F : \hat{K} \to A$, and every $\lambda \in \hat{K}$ induces an A-character $\mathcal{E}_\lambda : P(K, A) \to A$ given by $\mathcal{E}_\lambda(f) = F(\lambda)$. Thus the set of A-characters of $P(K, A)$ is in one-to-one correspondence with $\hat{K}$.

Example 4.3 (Vector-valued Lipschitz Algebras)**.** Let (X, ρ) be a compact metric space. An A-valued *Lipschitz function* is a function $f : X \to A$ such that

$$L(f) = \sup\left\{ \frac{\|f(x) - f(y)\|}{\rho(x, y)} : x, y \in X,\ x \neq y \right\} < \infty. \tag{4.1}$$

Denoted by $\mathrm{Lip}(X, A)$, the space of A-valued Lipschitz functions on X is an A-valued function algebra on X. For $f \in \mathrm{Lip}(X, A)$, the Lipschitz norm of f is defined by $\|f\|_L = \|f\|_X + L(f)$. It is easily verified that $(\mathrm{Lip}(X, A), \|\cdot\|_L)$ is an admissible Banach A-valued function algebra with $\mathrm{Lip}(X) = \mathrm{Lip}(X, A) \cap C(X)$, where $\mathrm{Lip}(X) = \mathrm{Lip}(X, \mathbb{C})$ is the classical complex-valued Lipschitz algebra. Recently, in [5], the character space and Šilov boundary of $\mathrm{Lip}(X, A)$ has been studied. Since $\mathfrak{A} = \mathrm{Lip}(X)$ is natural ([5] or [15]), the only A-characters of $\mathrm{Lip}(X, A)$ are the point evaluation homomorphisms $\mathcal{E}_x$ $(x \in X)$.

Next, let $\mathbb{T}$ be the unit circle in $\mathbb{C}$, and let $\mathrm{Lip}_P(\mathbb{T}, A)$ be the closure of $P_0(\mathbb{T}, A)$ in $\mathrm{Lip}(\mathbb{T}, A)$. Then $\mathfrak{A} = \mathrm{Lip}_P(\mathbb{T})$, the closure of $P_0(\mathbb{T})$ in $\mathrm{Lip}(\mathbb{T})$, with $\mathfrak{M}(\mathfrak{A}) = \Delta$, the closed unit disc. It is easily verified that $\|\hat{f}\| = \|f\|_{\mathbb{T}}$, for every $f \in \mathrm{Lip}(\mathbb{T})$. Hence, by Theorem 3.9, every $f \in \mathrm{Lip}_P(\mathbb{T}, A)$ extends to a function $F : \Delta \to A$, and every $\lambda \in \Delta$ induces an A-character $\mathcal{E}_\lambda : \mathrm{Lip}_P(\mathbb{T}, A) \to A$ given by $\mathcal{E}_\lambda(f) = F(\lambda)$. The set of A-characters of $\mathrm{Lip}_P(\mathbb{T}, A)$ is therefore in one-to-one correspondence with Δ.

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