Fixed points of Ćirić-Matkowski-type contractions in ν-generalized metric spaces

Mortaza Abtahi, Zoran Kadelburg and Stojan Radenović

Abstract. In this paper, fixed point theorems for Ćirić-Matkowski-type contractions in ν-generalized metric spaces are presented. Then, by replacing the distance function $d(x, y)$ with function of the form $m(x, y) = d(x, y) + \gamma\big(d(x, Tx) + d(y, Ty)\big)$, where $\gamma > 0$, results analogue to those due to P. D. Proinov [Fixed point theorems in metric spaces, Nonlinear Anal. 64 (2006) 546–557] are obtained. An example is provided to demonstrate a possible usage of these results.

Mathematics Subject Classification (2010). Primary 47H10; Secondary 54H25.

Keywords. ν-generalized metric space, Cauchy sequence, fixed point, Ćirić-Matkowski-type contraction.

1. Introduction and Preliminaries

Fixed point theory in metric spaces have many applications. It is natural that there have been several attempts to extend it to a more general setting. One of these generalizations was introduced by Branciari in 2000, where the triangle inequality was replaced by a so-called *polygonal inequality.* He introduced the concept of ν-generalized metric spaces as follows (see also [2, 5, 8, 9, 15, 16]).

Definiton 1.1 (Branciari [3]). Let X be a nonvoid set and $d\colon X \times X \to [0, \infty)$ be a mapping. Let $\nu \in \mathbb{N}$. Then (X, d) is called a *ν-generalized metric space* if the following hold:

1. $d(x, y) = 0$ if and only if $x = y$, for every $x, y \in X$;
2. $d(x, y) = d(y, x)$, for every $x, y \in X$;
3. $d(x, y) \le d(x, u_1) + d(u_1, u_2) + \cdots + d(u_\nu, y)$, for every set $\{x, u_1, \ldots, u_\nu, y\}$ of $\nu + 2$ elements of X that are all different.

Obviously, (X, d) is a metric space if and only if it is a 1-generalized metric space. In [15], it was shown that not every generalized metric space has a compatible topology.

Definiton 1.2 ([2]). Let (X,d) be a ν-generalized metric space. Let $k\in\mathbb{N}$. A sequence $\{x_n\}$ in X is said to be *k-Cauchy* if

$$\lim_{n\to\infty}\sup\{d(x_n,x_{n+1+mk}):m\in\mathbb{Z}^+\}=0. \tag{1.1}$$

The sequence $\{x_n\}$ is said to be *Cauchy* if it is 1-Cauchy.

The concept of Cauchy sequences in ν-generalized metric spaces was studied in [2, 16]; see also [3].

Proposition 1.3 ([2, 16]). *Let (X,d) be a ν-generalized metric space and let $\{x_n\}$ be a sequence in X such that x_n $(n\in\mathbb{N})$ are all different. Suppose that $\{x_n\}$ is ν-Cauchy. If ν is odd, or if ν is even and $d(x_n,x_{n+2})\to 0$ as $n\to\infty$, then $\{x_n\}$ is Cauchy.*

According to [3], a sequence $\{x_n\}$ in a ν-generalized metric space (X,d) is said to *converge* to x if $d(x,x_n)\to 0$ as $n\to\infty$. It was shown in [13] and [14] (see, e.g., [14, Example 1.1]) that, among other things, a sequence in a 2-generalized metric space may converge to more than one point and that a convergent sequence may not be a Cauchy sequence.

According to [2, 16], a sequence $\{x_n\}$ is said to *converge to x in the strong sense* if $\{x_n\}$ is Cauchy and $\{x_n\}$ converges to x. The mentioned [14, Example 1.1] shows that there exist convergent sequences in 2-generalized metric spaces that do not converge in the strong sense.

The space X is said to be *complete* if every Cauchy sequence in X converges. In [2], the completeness of ν-generalized metric spaces is discussed.

Proposition 1.4 ([16]). *Let $\{x_n\}$ and $\{y_n\}$ be sequences in a ν-generalized metric space (X,d) that converge to x and y in the strong sense, respectively. Then*

$$d(x,y)=\lim_{n\to\infty}d(x_n,y_n).$$

Branciari, in [3], proved a generalization of the Banach contraction principle. His proof was not fully correct because a ν-generalized metric space does not necessarily have the compatible topology, see [6, 13, 14, 15, 17]. A proof of the Banach contraction principle, as well as proofs of Kannan's [7] and Ćirić's [4] fixed point theorems, in ν-generalized metric spaces, can be found in [16].

Theorem 1.5 ([16]). *Let (X,d) be a complete ν-generalized metric space, and let T be a self-map of X. For every $x,y\in X$, let*

$$m(x,y)=\max\{d(x,y),d(x,Tx),d(y,Ty),d(x,Ty),d(y,Tx)\}. \tag{1.2}$$

Assume there exists $r\in[0,1)$ such that $d(Tx,Ty)\le rm(x,y)$, for all $x,y\in X$. Then T has a unique fixed point z and, moreover, for any $x\in X$, the Picard iterates T^nx $(n\in\mathbb{N})$ converge to z in the strong sense.

The present paper is organized as follows. In Section 2, we study Cauchy sequences in ν-generalized metric spaces. We present a necessary and sufficient condition for a sequence to be Cauchy. Next, in Section 3, we give new fixed point theorems in ν-generalized metric spaces. These results are extensions to ν-generalized metric spaces of the theorems by Meir and Keeler [11], Ćirić [4], Matkowski [10, Theorem 1.5.1], and Proinov [12]. It is shown by an example that these results are more powerful than some of the results from the paper [16].

Throughout the paper, the set of integers is denoted by $\mathbb{Z}$, the set of nonnegative integers is denoted by $\mathbb{Z}^+$, and the set of positive integers is denoted by $\mathbb{N}$.

2. Results on Cauchy Sequences

The following is the main result of the section.

Lemma 2.1. *Let $\{x_n\}$ be a sequence in a ν-generalized metric space (X, d) such that x_n $(n \in \mathbb{N})$ are all different. Suppose that, for every $\epsilon > 0$ and for any two subsequences $\{x_{p_i}\}$ and $\{x_{q_i}\}$, if $\limsup_{i\to\infty} d(x_{p_i}, x_{q_i}) \le \epsilon$, then, for some N,*

$$d(x_{p_i+1}, x_{q_i+1}) \le \epsilon \quad (i \ge N).$$

If, moreover, $d(x_n, x_{n+1}) \to 0$ as $n \to \infty$, then the sequence $\{x_n\}$ is ν-Cauchy.

Proof. Suppose $\{x_n\}$ is not ν-Cauchy. Then (1.1) fails to hold for $k = \nu$. Hence, there is $\epsilon > 0$ such that

$$\forall k \in \mathbb{N},\ \exists\, n \ge k, \quad \sup\{d(x_n, x_{n+1+m\nu}) : m \in \mathbb{Z}^+\} > \epsilon. \tag{2.1}$$

Since $d(x_n, x_{n+1}) \to 0$, there exist positive integers $k_1 < k_2 < \cdots$ such that

$$d(x_n, x_{n+1}) < \epsilon/i \quad (n \ge k_i).$$

For each k_i, by (2.1), there exist $n_i \ge k_i + 1$ and $m_i \in \mathbb{Z}^+$ such that

$$d(x_{n_i}, x_{n_i+1+m_i\nu}) > \epsilon.$$

Since $d(x_{n_i}, x_{n_i+1}) < \epsilon$, we have $m_i \ge 1$. We let m_i be the smallest number with this property so that $d(x_{n_i}, x_{n_i+1+m_i\nu-\nu}) \le \epsilon$. Now, let $p_i = n_i - 1$ and $q_i = n_i + m_i\nu$. Then $q_i > p_i \ge k_i$, and

$$d(x_{p_i+1}, x_{q_i+1}) > \epsilon, \quad d(x_{p_i+1}, x_{q_i+1-\nu}) \le \epsilon.$$

Using property (3) in Definition 1.1, since all x_n $(n \in \mathbb{N})$ are different, for every $i \in \mathbb{N}$, we have

$$\begin{aligned} d(x_{p_i}, x_{q_i}) \le d(x_{p_i}, x_{p_i+1}) &+ d(x_{p_i+1}, x_{q_i+1-\nu}) \\ &+ d(x_{q_i+1-\nu}, x_{q_i+2-\nu}) + \cdots + d(x_{q_i-1}, x_{q_i}). \end{aligned}$$

Therefore, $d(x_{p_i}, x_{q_i}) \le \nu\epsilon/i + \epsilon$, and thus $\limsup_{i\to\infty} d(x_{p_i}, x_{q_i}) \le \epsilon$. This is a contradiction, since $d(x_{p_i+1}, x_{q_i+1}) > \epsilon$, for all i. □

Theorem 2.2. *Suppose that $\{x_n\}$ satisfies all the conditions in Lemma 2.1, and, moreover, $d(x_n, x_{n+2}) \to 0$ as $n \to \infty$. Then $\{x_n\}$ is a Cauchy sequence.*

Proof. By Lemma 2.1, the sequence $\{x_n\}$ is ν-Cauchy. Since $d(x_n, x_{n+2}) \to 0$, by Proposition 1.3, the sequence $\{x_n\}$ is Cauchy. $\square$

Theorem 2.3. *Let $\{x_n\}$ be a sequence in a ν-generalized metric space (X, d) such that x_n $(n \in \mathbb{N})$ are all different and $d(x_n, x_{n+1}) + d(x_n, x_{n+2}) \to 0$ as $n \to \infty$. Assume that $m(x, y)$ is a nonnegative function on $X \times X$ such that, for any two subsequences $\{x_{p_i}\}$ and $\{x_{q_i}\}$,*

$$\limsup_{i\to\infty} m(x_{p_i}, x_{q_i}) \le \limsup_{i\to\infty} d(x_{p_i}, x_{q_i}). \tag{2.2}$$

The following condition then implies that the sequence $\{x_n\}$ is Cauchy: for every $\epsilon > 0$ and for any two subsequences $\{x_{p_i}\}$ and $\{x_{q_i}\}$, if $\limsup_{i\to\infty} m(x_{p_i}, x_{q_i}) \le \epsilon$, then, for some N,

$$d(x_{p_i+1}, x_{q_i+1}) \le \epsilon \quad (i \ge N). \tag{2.3}$$

Proof. Let $\epsilon > 0$ and let $\{x_{p_i}\}$ and $\{x_{q_i}\}$ be two subsequences with $\limsup_{i\to\infty} d(x_{p_i}, x_{q_i}) \le \epsilon$. By (2.2), we get $\limsup_{i\to\infty} m(x_{p_i}, x_{q_i}) \le \epsilon$. Therefore, (2.3) holds. All conditions in Lemma 2.1 are fulfilled and so the sequence is ν-Cauchy. Since $d(x_n, x_{n+1}) + d(x_n, x_{n+2}) \to 0$, by Proposition 1.3, we see that $\{x_n\}$ is a Cauchy sequence. $\square$

3. Fixed Point Theorems of Ćirić-Matkowski Type

Let (X, d) be a ν-generalized metric space. A mapping $T\colon X \to X$ is said to be a *Ćirić-Matkowski contraction* if $d(Tx, Ty) < d(x, y)$, for every $x, y \in X$ with $x \ne y$, and, for any $\epsilon > 0$, there exists $\delta > 0$ such that, for all $x, y \in X$,

$$d(x, y) < \delta + \epsilon \implies d(Tx, Ty) \le \epsilon. \tag{3.1}$$

The following lemma was proved in the context of metric spaces as [1, Lemma 3.1]. Its proof does not use the triangular inequality, so it holds true also in ν-generalized metric spaces.

Lemma 3.1. *Let (X, d) be a ν-generalized metric space. For a sequence $\{x_n\}$ in X and a nonnegative function $m(x, y)$ on $X \times X$, the following are equivalent:*

1. *for every $\epsilon > 0$, there exist $\delta > 0$ and $N \in \mathbb{Z}^+$ such that, for all $p, q \ge N$,*
$$m(x_p, x_q) < \epsilon + \delta \implies d(x_{p+1}, x_{q+1}) \le \epsilon. \tag{3.2}$$
2. *for every $\epsilon > 0$ and for any two subsequences $\{x_{p_i}\}$ and $\{x_{q_i}\}$, if $\limsup_{i\to\infty} m(x_{p_i}, x_{q_i}) \le \epsilon$ then, for some N, $d(x_{p_i+1}, x_{q_i+1}) \le \epsilon$ $(i \ge N)$.*

We will need the following lemma in the sequel.

Lemma 3.2. *Let (X, d) be a ν-generalized metric space and let $T : X \to X$ be a mapping. Suppose $d(T^n x, T^{n+1} x) \to 0$, as $n \to \infty$, for some $x \in X$. Then, for some $k \in \mathbb{N}$, either the Picard iterates $T^n x$ $(n \ge k)$ are all different or they are all the same.*

Proof. Suppose $T^{k+m}x = T^k x$, for some $k, m \in \mathbb{N}$, and let m be the smallest positive integer with this property. If $m = 1$, that is $T^{k+1}x = T^k x$, then $T^n x = T^k x$, for $n \geq k$, and there is nothing to prove. If $m \geq 2$, then every two elements in the set $\{T^k x, T^{k+1}x, \ldots, T^{k+m-1}x\}$ are different. Now, for $n > k$, write $n - k = mj + i$ with $j \geq 0$ and $0 \leq i \leq m - 1$. Then

$$d(T^n x, T^{n+1}x) = d(T^{k+mj+i}x, T^{k+mj+i+1}x) = d(T^{k+i}x, T^{k+i+1}x).$$

The above inequality contradicts the fact that $d(T^n x, T^{n+1}x) \to 0$. □

Now, suppose that T is a Ćirić-Matkowski contraction on X, take a point $x \in X$, and set $x_n = T^n x$ $(n \in \mathbb{N})$. Then, for every $\epsilon > 0$, there exist $\delta > 0$ such that $d(x_p, x_q) < \epsilon + \delta$ implies $d(x_{p+1}, x_{q+1}) \leq \epsilon$.

Theorem 3.3. *Let (X, d) be a ν-generalized metric space, let T be a self-map of X and let $m(x, y)$ be a nonnegative function on $X \times X$. Suppose that, for some point $x \in X$, the following conditions hold:*

1. *for any $\epsilon > 0$, there exist $\delta > 0$ and $N \in \mathbb{Z}^+$ such that, for all $p, q \geq N$,*
$$m(T^p x, T^q x) < \delta + \epsilon \implies d(T^{p+1}x, T^{q+1}x) \leq \epsilon, \tag{3.3}$$
2. *condition (2.2) holds for any two subsequences $\{T^{p_i}x\}$ and $\{T^{q_i}x\}$ of $\{T^n x\}$,*
3. $d(T^n x, T^{n+1}x) + d(T^n x, T^{n+2}x) \to 0$ *as $n \to \infty$.*

Then $\{T^n x\}$ is a Cauchy sequence.

Proof. Using Lemma 3.1, condition (3.3) implies that, for every $\epsilon > 0$ and for any two subsequences $\{T^{p_i}x\}$ and $\{T^{q_i}x\}$ of $\{T^n x\}$, if $\limsup_{i\to\infty} m(T^{p_i}x, T^{q_i}x) \leq \epsilon$ then, for some N, $d(T^{p_i+1}x, T^{q_i+1}x) \leq \epsilon$ $(i \geq N)$. By Lemma 3.2, the Picard iterates $T^n x$ are eventually all the same, in which case $\{T^n x\}$ is obviously a Cauchy sequence, or they are all different. In the latter case, Theorem 2.3 shows that the sequence $\{T^n x\}$ is Cauchy. □

Now we give a new proof of the result that appeared as [15, Theorem 13].

Theorem 3.4. *Let (X, d) be a complete ν-generalized metric space and let T be a Ćirić-Matkowski contraction on X. Then T has a unique fixed point z, and, moreover, for any $x \in X$, the sequence $\{T^n x\}$ converges to z in the strong sense.*

Proof. First, we show that T has at most one fixed point. Suppose $Tz = z$ and $y \neq z$. Then $d(Ty, Tz) = d(Ty, z) < d(y, z)$. Hence $Ty \neq y$.

Given $x \in X$, we consider the following two cases.

1. There exist $k, m \in \mathbb{N}$ such that $T^{k+m}x = T^k x$.
2. $T^n x$ $(n \in \mathbb{N})$ are all different.

In the case (a), where $T^{k+m}x = T^k x$, for some $k, m \in \mathbb{N}$, we let m be the smallest positive integer with this property. If $m = 1$, that is $T^{k+1}x = T^k x$, then $T^n x = T^k x$, for $n \geq k$, and there is nothing to prove. If $m \geq 2$, then every two successive elements in the following sequence are different:

$$T^k x, T^{k+1}x, \ldots, T^{k+m-1}x, T^{k+m}x, T^{k+m+1}x, \ldots$$

Recall that $x \neq y$ implies $d(Tx, Ty) < d(x, y)$. Hence

$$\begin{aligned} d(T^k x, T^{k+1}x) &= d(T^{k+m}x, T^{k+m+1}x) < d(T^{k+m-1}x, T^{k+m}x) \\ &< \cdots < d(T^{k+1}x, T^{k+2}x) < d(T^k x, T^{k+1}x). \end{aligned}$$

This is absurd.

In the case (b), we let $x_n = T^n x$, and show that $d(x_n, x_{n+i}) \to 0$, as $n \to \infty$, for $i = 1, 2$. Since x_n $(n \in \mathbb{N})$ are all different, we have $d(x_{n+1}, x_{n+i+1}) < d(x_n, x_{n+i})$, for every n, that is, the sequence $\epsilon_n = d(x_n, x_{n+i})$ is decreasing and thus $\epsilon_n \downarrow \epsilon$ for some $\epsilon \geq 0$. If $\epsilon > 0$, there is $\delta > 0$ such that $\epsilon_n = d(T^n x, T^{n+1}x) \leq \epsilon + \delta$ implies that $\epsilon_{n+1} = d(T^{n+1}x, T^{n+2}x) \leq \epsilon$. This is a contradiction since we have $\epsilon < \epsilon_n$, for all n. Hence, $d(x_n, x_{n+i}) \to 0$, as $n \to \infty$ $(i = 1, 2)$. Now, by Theorem 3.3, the sequence $\{T^n x\}$ is Cauchy. Since X is complete, $\{T^n x\}$ converges to some $z \in X$. By Proposition 1.4, we have

$$d(z, Tz) = \lim_{n\to\infty} d(T^n x, Tz) \leq \lim_{n\to\infty} d(T^{n-1}x, z) = 0.$$

Hence $Tz = z$, i.e., z is a fixed point of T. □

Similarly as in metric spaces, we will use the following terminology.

Definiton 3.5. A self-mapping T of a ν-generalized metric space (X, d) is said to be *sequentially continuous* if $\{Tx_n\}$ converges to Tx whenever $\{x_n\}$ converges to x. The mapping T is called *asymptotically regular* if

$$d(T^n x, T^{n+1}x) + d(T^n x, T^{n+2}x) \to 0, \text{ as } n \to \infty \quad (x \in X).$$

We are now in a position to state and prove a version of Proinov's theorem, [12, Theorem 4.2], for ν-generalized metric spaces.

Theorem 3.6. *Let (X, d) be a complete ν-generalized metric space, and T be a sequentially continuous and asymptotically regular self-map of X. For $\gamma > 0$, define m on $X \times X$ by $m(x, y) = d(x, y) + \gamma\big(d(x, Tx) + d(y, Ty)\big)$. Suppose that*

$$d(Tx, Ty) < m(x, y), \text{ for every } x, y \in X, \text{ with } x \neq y, \tag{3.4}$$

and that, for any $\epsilon > 0$, there exist $\delta > 0$ and $N \in \mathbb{N}_0$ such that, for all $x, y \in X$,

$$m(T^N x, T^N y) < \delta + \epsilon \implies d(T^{N+1}x, T^{N+1}y) \leq \epsilon. \tag{3.5}$$

Then T has a unique fixed point z, and, for any $x \in X$, the Picard iterates $T^n x$ $(n \in \mathbb{N})$ converge to z in the strong sense.

Proof. First, let us prove that T has at most one fixed point. If $Ty = y$ and $Tz = z$, then $m(y, z) = d(y, z) = d(Ty, Tz)$. Hence $y = z$ (otherwise, we should have $d(Ty, Tz) < d(y, z)$ which is not the case).

Now, choose $x \in X$ and set $x_n = T^n x$ $(n \in \mathbb{N})$. Since T is assumed to be asymptotically regular, we have $d(x_n, x_{n+1}) \to 0$ and $d(x_n, x_{n+2}) \to 0$ as $n \to \infty$. Hence, (2.2) holds for any two subsequences $\{x_{p_i}\}$ and $\{x_{q_i}\}$. By Theorem 2.3, the sequence $\{T^n x\}$ is Cauchy and, since X is complete, it converges to some point $z \in X$. Since T is sequentially continuous, we have $Tx_n \to Tz$. Since both $\{x_n\}$ and $\{Tx_n\}$ converge in the strong sense, by Proposition 1.4, we get

$$d(z, Tz) = \lim_{n\to\infty} d(x_n, Tx_n) = \lim_{n\to\infty} d(x_n, x_{n+1}) = 0.$$

Therefore, $Tz = z$ and z is the unique fixed point of T. $\square$

Example 3.7. Let $X = \{a, b, c, \delta, e\}$ and $d : X \times X \to [0, +\infty)$ be defined by:

$$\begin{aligned}
&d(x, x) = 0 \text{ for } x \in X;\\
&d(x, y) = d(y, x) \text{ for } x, y \in X;\\
&d(a, b) = 3,\\
&d(a, c) = d(b, c) = 1,\\
&d(a, \delta) = d(b, \delta) = d(c, \delta) = 2,\\
&d(a, e) = d(c, e) = 1,\ d(b, e) = d(\delta, e) = 2.
\end{aligned}$$

Then it is easy to check that (X, d) is a 2-generalized metric space which is not a metric space since

$$d(a, b) = 3 > 2 = d(a, c) + d(c, b).$$

Consider $T : X \to X$ given by

$$T = \begin{pmatrix} a & b & c & \delta & e \\ c & c & c & a & b \end{pmatrix}.$$

Then the mapping T is obviously sequentially continuous. Since, for each $x \in X$, $T^n x = c$ for n sufficiently large, it is clear that T is asymptotically regular and that condition (3.5) is fulfilled. Take $\gamma = 1$. In order to check the condition (3.4), it is nontrivial just to consider the following cases:

1. $x \in \{a, b, c\}$, $y \in \{\delta, e\}$. Then

$$d(Tx, Ty) = 1 < 3 \le m(x, y).$$

2. $\{x, y\} = \{\delta, e\}$, $x \neq y$. Then

$$d(Tx, Ty) = 3 < 6 = m(x, y).$$

Hence, all the conditions of Theorem 3.6 are fulfilled and T has a unique fixed point (which is $z = c$).

Note that for $x = \delta$, $y = e$ it is

$$d(Tx, Ty) = 3 > 2 = m(x, y).$$

Hence, the conditions of Theorem 1.5 do not hold and the conclusion cannot be reached using this result.

4. Conclusion

ν-generalized metric spaces were introduced by Branciari in [3] and some fixed point results were obtained. After that, several researchers proved various fixed point results in these spaces (in particular, improving some deductions from [3]), but mostly for the case $\nu = 2$. Some results for arbitrary ν were obtained in the papers [2, 15, 16]. We have extended some of these results in the present paper, in particular proving an analog of Proinov's fixed point result from [12] in the framework of ν-generalized metric spaces. It has been shown by an example that this result is more powerful than some of the results from the paper [16].

Acknowledgement

The second author is grateful to the Ministry of Education, Science and Technological Development of Serbia, grant No. 174002.

The authors express their sincere gratitude to the anonymous referees for their careful reading and suggestions that improved the presentation of this paper.

References

[1] M. Abtahi, *Fixed point theorems for Meir-Keeler type contractions in metric spaces*, Fixed Point Theory (to appear).

[2] B. Alamri, T. Suzuki and L.A. Khan, *Caristi's fixed point theorem and Subrahmanyam's fixed point theorem in ν-generalized metric spaces*, J. Function Spaces, 2015, Art. ID 709391, 6 pp.

[3] A. Branciari, *A fixed point theorem of Banach-Caccioppoli type on a class of generalized metric spaces*, Publ. Math. Debrecen, 57 (2000), 31–37.

[4] Lj. B. Ćirić, *A new fixed-point theorem for contractive mappings*, Publ. Inst. Math. (N.S) 30 (44) (1981), 25–27.

[5] Z. Kadelburg and S. Radenović, *On generalized metric spaces: A survey*, TWMS J. Pure Appl. Math., 5 (2014), 3–13.

[6] Z. Kadelburg and S. Radenović, *Fixed point results in generalized metric spaces without Hausdorff property*, Math. Sciences, 8:125, 2014.

[7] R. Kannan, *Some results on fixed points-II*, Amer. Math. Monthly, 76 (1969), 405–408.

[8] W. A. Kirk and N. Shahzad, *Generalized metrics and Caristi's theorem*, Fixed Point Theory Appl., 2013, 2013:129.

[9] W. A. Kirk and N. Shahzad, *Corrections: Generalized metrics and Caristi's theorem*, Fixed Point Theory Appl., 2014, 2014:177.

[10] M. Kuczma, B. Choczewski and R. Ger, *Iterative functional equations, Encyclopedia of Mathematics and Applications*, vol. 32, Cambridge University Press, Cambridge, 1990.

[11] A. Meir, E. Keeler, *A theorem on contraction mappings*, J. Math. Anal. Appl., 28 (1969), 326–329.

[12] P.D. Proinov, *Fixed point theorems in metric spaces*, Nonlinear Anal. 64 (2006), 546–557.

[13] B. Samet, *Discussion on 'A fixed point theorem of Banach-Caccioppoli type on a class of generalized metric spaces' by A. Branciari*, Publ. Math. Debrecen, 76 (2010), 493–494.

[14] I.R. Sarma, J.M. Rao and S.S. Rao, *Contractions over generalized metric spaces*, J. Nonlinear Sci. Appl. 2 (2009), 180–182.

[15] T. Suzuki, *Generalized metric spaces do not have the compatible topology*, Abstr. Appl. Anal., 2014, Art. ID 458098, 5 pp.

[16] T Suzuki, B Alamri and L.A. Khan, *Some notes on fixed point theorems in ν-generalized metric spaces*, Bull. Kyushu Inst. Tech. Pure Appl. Math. 62 (2015), 15–23.

[17] M. Turinici, *Functional contractions in local Branciari metric spaces*, Romai J. 8 (2012), 189–199.

Mortaza Abtahi
School of Mathematics and Computer Sciences, Damghan University
Damghan, P.O.B. 36715-364, Iran
e-mail: `abtahi@du.ac.ir`

Zoran Kadelburg
(corresponding author)
Faculty of Mathematics, University of Belgrade
Studentski trg 16, 11000 Beograd, Serbia
e-mail: `kadelbur@matf.bg.ac.rs`

Stojan Radenović
Faculty of Mathematics and Information Technology, Teacher Education, Dong Thap University
Cao Lanch City, Dong Thap Province, Viet Nam
e-mail: `fixedpoint50@gmail.com`